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Bootstrap equations and correlation functions for the Heisenberg XYZ antiferromagnet

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Abstract

We present two kinds of integral solutions to the quantum Knizhnik–Zamolodchikov equations for the $2n$-point correlation functions of the Heisenberg XYZ antiferromagnet. Our first integral solution can be obtained from those for the cyclic SOS model by using the vertex–face correspondence. By the construction, the sum with respect to the local height variables $k_0, k_1, \ldots, k_{2n}$ of the cyclic SOS model remains other than the n-fold integral in the first solution. In order to perform these summations, we improve this to find the second integral solution of $(r+1)n$-fold integral for $r \in \mathbb{Z}_{>1}$, where r is a parameter of the XYZ model. Furthermore, we discuss the relations among our formula, Lashkevich–Pugai's formula and that of Shiraishi.

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1. Introduction

Thirty years have passed since Baxter's papers on the eight-vertex model [1] and the XYZ spin chain [2] were published. In a series of papers [3–5] Baxter constructed the eigenvectors of the Hamiltonian of the XYZ spin chain by translating the eight-vertex model to the equivalent face model, the eight-vertex SOS model.

Though many authors have tried to extract the results on correlation functions of the eight-vertex/XYZ model, very few results have been obtained. The spontaneous polarization was conjectured by Baxter and Kelland [6], and the same results were reproduced by solving a set of difference equations, the quantum Knizhnik–Zamolodchikov equation (q-KZ equations) in [7]. On the basis of the Z-invariance the integral formulae of correlation functions were obtained in the special case that the eight-vertex model can be factorized into two independent Ising models [8]. A free field representation of the type I [9] and type II [10] vertex operators was constructed to express the correlation functions and the form factors of the eight-vertex/XYZ model in terms of those of the eight-vertex SOS model [11].

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There are several ways of addressing the problem of correlation functions of integrable models. The vertex operator approach [12] provides a powerful tool to calculate the correlation functions of the six-vertex/XXZ model [13], the RSOS-type model [11, 14, 15]. The bootstrap approach [7, 16] is based on the corner transfer matrix (CTM) method [17] and enables us to derive a set of difference equations of correlation functions. Note that the CTM method can be applied to a massive integrable model such as the XYZ Heisenberg chain in the antiferromagnetic regime. Kitanin *et al* [18, 19] developed the algebraic Bethe ansatz method for obtaining domain wall correlation functions of the XXZ model, using the determinant structure of the partition function.

In this paper we try to construct the correlation functions of the XYZ antiferromagnet by directly solving a set of difference equations, bootstrap equations. These equations are derived on the basis of the CTM bootstrap in [7]. In a previous paper [20] we did the same thing for both the bulk and boundary XXZ antiferromagnets. Let us cite some results concerning the bulk XXZ case from [20].

The $2n$-point correlation functions of the bulk XXZ antiferromagnet with the spectral parameters $(\zeta) = (\zeta_1, \dots, \zeta_{2n})$ and the spin variables $\varepsilon = (\varepsilon_1, \dots, \varepsilon_{2n})$ can be expressed up to a scalar function of ζ in the following form:

$$\overline{G}_\sigma^{(n)}(\zeta)^\varepsilon = \prod_{a\in A} \oint_{C_a} \frac{\mathrm{d}w_a}{2\pi\sqrt{-1}w_a} \Psi_\sigma^{(n)}(\{w_a\}_{a\in A}|\zeta) Q^{(n)}(\{w_a\}_{a\in A}|\zeta)^\varepsilon \qquad (\sigma = \pm) \tag{1.1}$$

where

$$A := \{a|\varepsilon_a > 0, 1 \leqslant a \leqslant 2n\}. \tag{1.2}$$

The function $Q^{(n)}(w|\zeta)^\varepsilon$ is a meromorphic function defined below, and $\Psi_\sigma^{(n)}(w|\zeta)$ is a kernel with appropriate transformation properties [20, equations (2.32)–(2.33)] and recursion relations [20, equation (2.34)]. The subscript $\sigma = \pm$ in (1.1) specifies one of the two vacuums of the XXZ model in the antiferromagnetic regime. The explicit expression of $Q^{(n)}(w|\zeta)^\varepsilon$ is as follows:

$$Q^{(n)}(w|\zeta)^\varepsilon = \prod_{a\in A} \zeta_a \left(\prod_{j=1}^{a-1} (xz_j - w_a) \prod_{j=a+1}^{2n} (xw_a - z_j) \right) \prod_{\substack{a,b\in A\\ a<b}} (x^{-1}w_b - xw_a)^{-1} \tag{1.3}$$

where $z_j = \zeta_j^2$. Here we note the relation between our previous work [20] and [21, 22], the latter in which the correlation functions of the massless XXZ model were obtained. Since the XXZ model has only one vacuum in the massless regime, the vacuum structure of the massless XXZ model is very different from that of the XXZ antiferromagnet. Nevertheless, the structures of the integral formulae for the correlation functions in both regimes are quite similar. In order to see such similarity, let us substitute $x = \mathrm{e}^{-\epsilon}$, $\zeta_j = \mathrm{e}^{-\epsilon\beta_j}$ and $w_a = \mathrm{e}^{-2\epsilon\alpha_a}$ into (1.3). Then we have

$$\begin{aligned} Q^{(n)}(w|\zeta)^\varepsilon = \mathrm{const} \prod_{a\in A} \mathrm{e}^{(2n-1)\epsilon\alpha_a} \left(\prod_{j=1}^{a-1} \mathrm{sh}\, \epsilon \left(\alpha_a - \beta_j - \frac{1}{2}\right) \prod_{j=a+1}^{2n} \mathrm{sh}\, \epsilon \left(\beta_j - \alpha_a - \frac{1}{2}\right) \right) \\ \times \prod_{j=1}^{2n} \mathrm{e}^{n\epsilon\beta_j} \prod_{\substack{a,b\in A\\ a<b}} (\mathrm{sh}\, \epsilon(\alpha_a - \alpha_b + 1))^{-1}. \end{aligned} \tag{1.4}$$

When $|x| = 1$ ($\epsilon \in \sqrt{-1}\mathbb{R}$), expression (1.4) is equal to the meromorphic function $Q_n(\alpha|\beta)^\varepsilon$ in [21], the massless analogue of $Q^{(n)}(w|\zeta)^\varepsilon$, up to a trivial factor. Furthermore, we should note here that the corresponding expression for the eight-vertex SOS model which will be given in section 2 also possesses quite similar structure to (1.4).

The XYZ Heisenberg antiferromagnet has two parameters q and p, where $p = (-q)^{2r}$. Lashkevich and Pugai [9] gave the n-fold integral formulae for the $2n$-point correlation functions G in terms of those of the SOS model F as follows:

$$G(u_1, \ldots, u_{2n}|u_0) = \sum_{k_0,k_1,\ldots,k_{2n}} t^{k_0}_{k_1}(u_1 - u_0) \otimes \cdots \otimes t^{k_{2n-1}}_{k_{2n}}(u_{2n} - u_0) F(u_1, \ldots, u_{2n}|u_0)^{k_0k_1\cdots k_{2n}}. \tag{1.5}$$

Here $u_1, \ldots, u_{2n}$ are additive spectral parameters, u_0 is an auxiliary parameter and $t^k_{k\pm1}(u)$ are the intertwining vectors introduced by Baxter [3, 4]. The sum with respect to k_j $(1 \leqslant j \leqslant 2n)$ should be taken over $k_j = k_{j-1} \pm 1$. However, the sum with respect to k_0 depends on the value of r. In (1.5) $r > 1$ is supposed. When r is a rational number of the form $2r = N/N'$ (N, N' coprime), we should take the sum over $\mathbb{Z}/N\mathbb{Z}$, otherwise over $\mathbb{Z}$.

On the other hand, Shiraishi [23] constructed the formulae of the correlation functions of the XYZ model without using the vertex–face correspondence. There are three main differences between Lashkevich–Pugai's formulae and those of Shiraishi. Shiraishi restricted himself to the case $r = \frac{3}{2}$, while $r > 1$ in (1.5). Furthermore, the $2n$-point correlation functions in the style of Shiraishi are of the $2n$-fold integral form, whilst being n-fold in the former case. These two differences seem to be the disadvantages of Shiraishi's formulae in a sense; however, the third difference gives an advantage. Shiraishi constructed the type II and type I vertex operators of the eight-vertex model as intertwiners of the q-deformed Virasoro algebra [24], without transforming those of the SOS model via vertex–face correspondence. Thus, the sums with respect to k_j in (1.5) are not needed.

In the present paper we wish to synthesize the advantages of the two formulae. Namely, we construct $(r + 1)n$-fold integral formulae for the $2n$-point correlation of the XYZ model with $r \in \mathbb{Z}_{>1}$, by using the vertex–face correspondence, and performing the sum with respect to k_j.

The rest of the present paper is organized as follows. In section 2 we formulate the XYZ antiferromagnet and the corresponding cyclic SOS model. In section 3 we present the bootstrap equations and construct integral formulae for correlation functions for the cyclic SOS model. In section 4 we further present integral formulae for the correlation functions of the XYZ antiferromagnet. In section 5 we give some concluding remarks. In the appendix we list some properties of the R-matrix of the eight-vertex model.

2. The XYZ antiferromagnet and the cyclic SOS model

2.1. The Hamiltonian and the R-matrix

The Jacobi theta functions with the characteristics $a, b \in \mathbb{R}$ are defined by

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(u; \tau) := \sum_{m\in\mathbb{Z}} \exp\{\pi\sqrt{-1}(m + a)[(m + a)\tau + 2(u + b)]\}. \tag{2.1}$$

The ith theta functions are defined as follows:

$$\theta_1(u; \tau) = \vartheta\begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}(u; \tau) \qquad \theta_2(u; \tau) = \vartheta\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}(u; \tau)$$
$$\theta_3(u; \tau) = \vartheta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(u; \tau) \qquad \theta_4(u; \tau) = \vartheta\begin{bmatrix} 0 \\ 1/2 \end{bmatrix}(u; \tau).$$

These theta functions have the infinite product forms, e.g.,

$$\theta_1(u;\tau) = \sqrt{-1}q^{\frac{1}{4}}\,\mathrm{e}^{-\sqrt{-1}\pi u}\Theta_{q^2}(\mathrm{e}^{2\sqrt{-1}\pi u}) \qquad \theta_2(u;\tau) = q^{\frac{1}{4}}\,\mathrm{e}^{-\sqrt{-1}\pi u}\Theta_{q^2}(-\mathrm{e}^{2\sqrt{-1}\pi u})$$

where $q = \exp(\sqrt{-1}\pi\tau)$ and we used the standard notation

$$\Theta_p(z) := (z;p)_\infty(pz^{-1};p)_\infty(p,p)_\infty \qquad (a;p_1,\ldots,p_n)_\infty = \prod_{k_i\geqslant 0}\left(1-ap_1^{k_1}\cdots p_n^{k_n}\right).$$

In this section we consider the XYZ spin chain in an infinite lattice [5, 17]

$$H_{\mathrm{XYZ}} = -\frac{1}{2}\sum_{j\in\mathbb{Z}}\left(\sigma^x_{j+1}\sigma^x_j + \Gamma\sigma^y_{j+1}\sigma^y_j + \Delta\sigma^z_{j+1}\sigma^z_j\right). \tag{2.2}$$

Here σ^x_j, σ^y_j and σ^z_j denote the standard Pauli matrices acting on the jth site, and we restrict ourselves to the antiferromagnetic regime $|\Gamma| < 1$ and $\Delta < -1$, where

$$\Gamma = \frac{1-\gamma}{1+\gamma} \qquad \gamma = -\frac{\theta_1^2\left(\frac{\sqrt{-1}\epsilon}{\pi};\frac{2\sqrt{-1}\epsilon r}{\pi}\right)}{\theta_4^2\left(\frac{\sqrt{-1}\epsilon}{\pi};\frac{2\sqrt{-1}\epsilon r}{\pi}\right)} > 0$$

$$\Delta = -\frac{1}{1+\gamma}\frac{\theta_4^2\left(0;\frac{2\sqrt{-1}\epsilon r}{\pi}\right)(\theta_2\theta_3)\left(\frac{\sqrt{-1}\epsilon}{\pi};\frac{2\sqrt{-1}\epsilon r}{\pi}\right)}{(\theta_2\theta_3)\left(0;\frac{2\sqrt{-1}\epsilon r}{\pi}\right)\theta_4^2\left(\frac{\sqrt{-1}\epsilon}{\pi};\frac{2\sqrt{-1}\epsilon r}{\pi}\right)}.$$

The condition $|\Gamma| < 1$ and $\Delta < -1$ is equivalent to the one such that $\epsilon > 0$ and $r > 1$. The antiferromagnetic XXZ Hamiltonian can be obtained by taking the limit $r \to \infty$. In this limit we have $\gamma \to 0$ and $\Delta \to -(x + x^{-1})/2$, where $0 < x = \mathrm{e}^{-\epsilon} < 1$. As is well known, the XXZ Hamiltonian thus obtained commutes with the quantum affine algebra $U_{-x}(\widehat{\mathfrak{sl}_2})$ [12]. Let $V = \mathbb{C}v_+ + \mathbb{C}v_-$ be a vector representation of $U_{-x}(\widehat{\mathfrak{sl}_2})$. Then the limiting XXZ Hamiltonian formally acts on $V^{\otimes\infty} = \cdots\otimes V\otimes V\otimes\cdots$. In [12] the space of states $V^{\otimes\infty}$ was identified with the tensor product of level 1 highest and level -1 lowest representations of $U_{-x}(\widehat{\mathfrak{sl}_2})$. It is very likely that the same structure survives in the nonlimiting generic case. In other words, the XYZ antiferromagnet is expected to have the symmetry described by the elliptic affine algebra $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}_2})$ [25, 26], where $q = -x$ and $p = x^{2r}$.

The XYZ Hamiltonian (2.2) can be obtained from the transfer matrix for the eight-vertex model, by taking the logarithmic derivative with respect to the spectral parameter ζ. The R-matrix $R_{8V}(\zeta) \in \mathrm{End}(\mathbb{C}^2\otimes\mathbb{C}^2)$ of the eight-vertex model is given as follows:

$$\begin{aligned} R_{8V}(\zeta) &= \frac{1}{\kappa(\zeta)}\overline{R}_{8V}(\zeta) \\ \overline{R}_{8V}(\zeta) &= w_4(u)I\otimes I + w_2(u)\sigma^x\otimes\sigma^x + w_3(u)\sigma^y\otimes\sigma^y + w_1(u)\sigma^z\otimes\sigma^z \end{aligned} \tag{2.3}$$

where $z = \zeta^2 = x^{2u}$ and

$$w_\alpha(u) = \frac{\theta_\alpha\left(\frac{1}{2r}+\frac{u}{r};\frac{\pi\sqrt{-1}}{\epsilon r}\right)}{\theta_\alpha\left(\frac{1}{2r};\frac{\pi\sqrt{-1}}{\epsilon r}\right)}.$$

In what follows we denote $R_{8V}(\zeta)$ and $\overline{R}_{8V}(\zeta)$ by $R(\zeta)$ and $\overline{R}(\zeta)$, respectively, when there is no fear of confusion. In the above expression we modify Sklyanin's parametrization [27] such that (2.3) coincides with that of Baxter [17] in the principal regime. The normalization factor

$$\begin{aligned} \kappa(\zeta) &= \frac{[u+1]}{[1]}\bar{\kappa}(\zeta) \qquad [u] = x^{\frac{u^2}{r}-u}\Theta_{x^{2r}}(x^{2u}) \\ \bar{\kappa}(\zeta) &= \zeta^{\frac{r-1}{r}}\frac{(x^4z;x^4,x^{2r})_\infty(x^2z^{-1};x^4,x^{2r})_\infty(x^{2r}z;x^4,x^{2r})_\infty(x^{2r+2}z^{-1};x^4,x^{2r})_\infty}{(x^4z^{-1};x^4,x^{2r})_\infty(x^2z;x^4,x^{2r})_\infty(x^{2r}z^{-1};x^4,x^{2r})_\infty(x^{2r+2}z;x^4,x^{2r})_\infty} \end{aligned} \tag{2.4}$$

is chosen such that the partition function per site is unity. In other words, the factor $\kappa(\zeta)$ is the partition function per site of the unnormalized model defined by $\overline{R}(\zeta)$. For later convenience we also introduce the symbol $\{u\}$ by

$$\{u\} = x^{\frac{u^2}{r}-u}\Theta_{x^{2r}}(-x^{2u}).$$

Note that

$$\theta_1\left(\frac{u}{r};\frac{\pi\sqrt{-1}}{\epsilon r}\right) = \sqrt{\frac{\epsilon r}{\pi}}\exp\left(-\frac{\epsilon r}{4}\right)[u] \qquad \theta_4\left(\frac{u}{r};\frac{\pi\sqrt{-1}}{\epsilon r}\right) = \sqrt{\frac{\epsilon r}{\pi}}\exp\left(-\frac{\epsilon r}{4}\right)\{u\}.$$

Let $\mathbb{C}^2 = \mathbb{C}v_+ \oplus \mathbb{C}v_-$ and introduce the matrix elements of the R-matrix as follows:

$$R(\zeta)v_{\varepsilon_1}\otimes v_{\varepsilon_2} = \sum_{\varepsilon_1',\varepsilon_2'=\pm} v_{\varepsilon_1'}\otimes v_{\varepsilon_2'}R(\zeta)_{\varepsilon_1\varepsilon_2}^{\varepsilon_1'\varepsilon_2'} = \frac{1}{\kappa(\zeta)}\sum_{\varepsilon_1',\varepsilon_2'=\pm} v_{\varepsilon_1'}\otimes v_{\varepsilon_2'}\overline{R}(\zeta)_{\varepsilon_1\varepsilon_2}^{\varepsilon_1'\varepsilon_2'}. \tag{2.5}$$

Then the nonzero entries are given by

$$\begin{aligned}
\overline{R}(\zeta)_{++}^{++} = \overline{R}(\zeta)_{--}^{--} = a(\zeta) &= \frac{(\theta_2\theta_3)\left(\frac{u}{2r};\frac{\pi\sqrt{-1}}{\epsilon r}\right)(\theta_1\theta_4)\left(\frac{u+1}{2r};\frac{\pi\sqrt{-1}}{\epsilon r}\right)}{(\theta_2\theta_3)\left(0;\frac{\pi\sqrt{-1}}{\epsilon r}\right)(\theta_1\theta_4)\left(\frac{1}{2r};\frac{\pi\sqrt{-1}}{\epsilon r}\right)}\\
\overline{R}(\zeta)_{+-}^{+-} = \overline{R}(\zeta)_{-+}^{-+} = b(\zeta) &= -\frac{(\theta_1\theta_4)\left(\frac{u}{2r};\frac{\pi\sqrt{-1}}{\epsilon r}\right)(\theta_2\theta_3)\left(\frac{u+1}{2r};\frac{\pi\sqrt{-1}}{\epsilon r}\right)}{(\theta_2\theta_3)\left(0;\frac{\pi\sqrt{-1}}{\epsilon r}\right)(\theta_1\theta_4)\left(\frac{1}{2r};\frac{\pi\sqrt{-1}}{\epsilon r}\right)}\\
\overline{R}(\zeta)_{-+}^{+-} = \overline{R}(\zeta)_{+-}^{-+} = c(\zeta) &= \frac{(\theta_2\theta_3)\left(\frac{u}{2r};\frac{\pi\sqrt{-1}}{\epsilon r}\right)(\theta_2\theta_3)\left(\frac{u+1}{2r};\frac{\pi\sqrt{-1}}{\epsilon r}\right)}{(\theta_2\theta_3)\left(0;\frac{\pi\sqrt{-1}}{\epsilon r}\right)(\theta_2\theta_3)\left(\frac{1}{2r};\frac{\pi\sqrt{-1}}{\epsilon r}\right)}\\
\overline{R}(\zeta)_{--}^{++} = \overline{R}(\zeta)_{++}^{--} = d(\zeta) &= -\frac{(\theta_1\theta_4)\left(\frac{u}{2r};\frac{\pi\sqrt{-1}}{\epsilon r}\right)(\theta_1\theta_4)\left(\frac{u+1}{2r};\frac{\pi\sqrt{-1}}{\epsilon r}\right)}{(\theta_2\theta_3)\left(0;\frac{\pi\sqrt{-1}}{\epsilon r}\right)(\theta_2\theta_3)\left(\frac{1}{2r};\frac{\pi\sqrt{-1}}{\epsilon r}\right)}.
\end{aligned} \tag{2.6}$$

Assume that the parameters u, ϵ and r lie in the so-called principal regime [17]:

$$\epsilon > 0 \qquad r > 1 \qquad -1 < u < 0 \qquad 0 < p < x < \zeta^{-1} < 1$$

where $x = \mathrm{e}^{-\epsilon}$, $p = x^{2r}$ and $\zeta = x^u$. The main properties of the R-matrix are listed in the appendix.

2.2. Vertex–face correspondence

In order to construct the eigenvectors of the eight-vertex model, Baxter introduced the following intertwining vector [3, 4]:

$$\begin{aligned}
t_{k\pm1}^k(u) &= f(u)\bar{t}_{k\pm1}^k(u)\\
\bar{t}_{k\pm1}^k(u) &= (\sqrt{-1})^k(\theta_1\theta_4)\left(\frac{k\mp u}{2r};\frac{\pi\sqrt{-1}}{\epsilon r}\right)v_+ + (\sqrt{-1})^{-k}(\theta_2\theta_3)\left(\frac{k\mp u}{2r};\frac{\pi\sqrt{-1}}{\epsilon r}\right)v_-.
\end{aligned} \tag{2.7}$$

The normalization factor $f(u)$ satisfies the relation

$$C^2[u]f(u)f(u-1) = 1 \qquad C = \frac{\epsilon r}{\pi}\,\mathrm{e}^{-\frac{\epsilon r}{4}}\frac{(x^{2r},x^{2r})_\infty}{(-x^{2r},x^{2r})_\infty}. \tag{2.8}$$

The explicit expression of $f(u)$ is as follows:

$$f(u) = \frac{x^{-\frac{u^2}{2r}+\frac{r-1}{2r}u+\frac{1}{4}}}{C\sqrt{(x^{2r};x^{2r})_\infty}}\frac{(x^4z;x^4,x^{2r})_\infty(x^{2r+2}z;x^4,x^{2r})_\infty}{(x^2z;x^4,x^{2r})_\infty(x^{2r}z;x^4,x^{2r})_\infty}. \tag{2.9}$$

The SOS model is a face model [4] which is defined on the square lattice with a site variable $k_i \in \mathbb{Z}$ attached to each site i. We call k_i local state or height and impose the condition that the heights of the adjoining sites differ by one. Local Boltzmann weight is assumed to be a function of the spectral parameter u and to be denoted by $W\left(\begin{smallmatrix} c & d \\ b & a \end{smallmatrix}\middle| u\right)$, which is given for a state configuration ${}^{c}_{b}\square^{d}_{a}$ around a face. Here the four states a, b, c and d are ordered clockwise from the SE corner. The nonzero Boltzmann weights are given as follows:

$$\begin{aligned}
W\left(\begin{matrix} k\pm 2 & k\pm 1 \\ k\pm 1 & k \end{matrix}\middle| u\right) &= \frac{1}{\bar{\kappa}(u)} \\
W\left(\begin{matrix} k & k\pm 1 \\ k\pm 1 & k \end{matrix}\middle| u\right) &= \frac{1}{\bar{\kappa}(u)} \frac{[1]\{k\mp u\}}{[u+1]\{k\}} \\
W\left(\begin{matrix} k & k\mp 1 \\ k\pm 1 & k \end{matrix}\middle| u\right) &= -\frac{1}{\bar{\kappa}(u)} \frac{[u]\{k\pm 1\}}{[u+1]\{k\}}.
\end{aligned} \tag{2.10}$$

The normalization factor $\bar{\kappa}(u)$ given by (2.4) is chosen such that the partition function per site is unity. Then we have the so-called vertex–face correspondence [4]:

$$R(u_1-u_2)t^c_b(u_1)\otimes t^b_a(u_2) = \sum_d W\left(\begin{matrix} c & d \\ b & a \end{matrix}\middle| u_1-u_2\right) t^d_a(u_1)\otimes t^c_d(u_2). \tag{2.11}$$

The RHSs of the last two equations in (2.10) contain the factors such as $\{k\}$, while the corresponding Boltzmann weights of the ABF model [28] are expressed in terms of the factors such as $[k]$. Suppose that r is a positive integer greater than 1. Then the ABF model is called a restricted SOS (RSOS) model, since $[k]=0$ for $k\in r\mathbb{Z}$ and, therefore, the local states can be restricted as $k=1,\ldots,r-1$. On the other hand, the present face model (2.10) is a cyclic SOS model because $\{k+r\}=\{k\}$. In [9] the RSOS-type weights were used so that the following regularization was required. In order to avoid the pole resulting from $[k]$ in the denominator, $k\in\mathbb{Z}+\delta$ should be assumed with some real δ, and the limit $\delta\to 0$ should be taken after all calculation [9]. In our case we do not need such regularization because $\{k\}\neq 0$ for $k\in\mathbb{Z}$.

We also note that the intertwining vector (2.7) is closely connected with the $2r$-dimensional cyclic representation of Sklyanin algebra [29, 30]. Actually, the L-operator defined via $R_{8V}LL=LLR_{8V}$ relation can be factorized into the intertwining vector (2.7) and its dual vector.

3. Bootstrap equations in the cyclic SOS model

3.1. Integral formulae for the cyclic SOS case

As a preliminary, let us consider the bootstrap equations and correlation functions for the cyclic SOS model. Let

$$F^{(n)}_\sigma(\zeta_1,\ldots,\zeta_{2n})^{kk_1\cdots k_{2n-1}k} \qquad (\sigma=\pm) \tag{3.1}$$

for $|k_j-k_{j-1}|=1$ $(1\leqslant j\leqslant 2n)$ and $k_0=k_{2n}=k$, be the $2n$-point correlation function of the eight-vertex SOS model. Then these functions satisfy the following three CTM bootstrap equations [16]:

$$\begin{aligned}
&F^{(n)}_\sigma(\ldots,\zeta_{j+1},\zeta_j,\ldots)^{\cdots k_{j-1}k_jk_{j+1}\cdots} \\
&\qquad = \sum_{k'_j} W\left(\begin{matrix} k_{j+1} & k_j \\ k'_j & k_{j-1} \end{matrix}\middle| \zeta_j/\zeta_{j+1}\right) F^{(n)}_\sigma(\ldots,\zeta_j,\zeta_{j+1},\ldots)^{\cdots k_{j-1}k'_jk_{j+1}\cdots}
\end{aligned} \tag{3.2}$$

$$F_\sigma^{(n)}(\zeta_1,\ldots,\zeta_{2n-1},x^2\zeta_{2n})^{k\cdots k_{2n-1}k}=\sigma\frac{\{k\}}{\{k_{2n-1}\}}F_\sigma^{(n)}(\zeta_{2n},\zeta_1,\ldots,\zeta_{2n-1})^{k_{2n-1}k\cdots k_{2n-1}}. \tag{3.3}$$

$$F_\sigma^{(n)}(\zeta_1,\ldots,\zeta_{2n-2},\zeta_{2n-1},x^{-1}\zeta_{2n-1})^{k\cdots k_{2n-2}k_{2n-1}k}=\frac{\delta_{k_{2n-2},k}}{\{k\}}F_\sigma^{(n-1)}(\zeta_1,\ldots,\zeta_{2n-2})^{k\cdots k}. \tag{3.4}$$

$$F_\sigma^{(n)}(\zeta_1,\ldots,\zeta_{2n-2},\zeta_{2n-1},-x^{-1}\zeta_{2n-1})^{k\cdots k_{2n-2}k\pm1k}$$
$$=\delta_{k_{2n-2},k}\frac{\mathrm{e}^{-\frac{\pi\sqrt{-1}}{2r}(1\pm2k)}}{\sqrt{-1}\{k\}}F_\sigma^{(n-1)}(\zeta_1,\ldots,\zeta_{2n-2})^{k\cdots k}. \tag{3.5}$$

Set

$$F_\sigma^{(n)}(\zeta)^{kk_1\cdots k_{2n-1}k}=c_n\prod_{1\leqslant j<k\leqslant 2n}\zeta_j^{\frac{r-1}{r}}g(z_j/z_k)\times\overline{F}_\sigma^{(n)}(\zeta)^{kk_1\cdots k_{2n-1}k}. \tag{3.6}$$

Here c_n is a constant which will be determined below, and the function $g(z)$ has the properties

$$g(z)=g(x^{-4}z^{-1})\qquad \bar{\kappa}(\zeta)=\zeta^{\frac{r-1}{r}}\frac{g(z)}{g(z^{-1})}. \tag{3.7}$$

The explicit form of $g(z)$ is as follows:

$$g(z)$$
$$=\frac{(x^6z;x^4,x^4,x^{2r})_\infty(x^2z^{-1};x^4,x^4,x^{2r})_\infty(x^{2r+6}z;x^4,x^4,x^{2r})_\infty(x^{2r+2}z^{-1};x^4,x^4,x^{2r})_\infty}{(x^8z;x^4,x^4,x^{2r})_\infty(x^4z^{-1};x^4,x^4,x^{2r})_\infty(x^{2r+4}z;x^4,x^4,x^{2r})_\infty(x^{2r}z^{-1};x^4,x^4,x^{2r})_\infty}. \tag{3.8}$$

In order to present our integral formulae for $\overline{F}_\sigma^{(n)}(\zeta)$ let us prepare some notation. Let

$$A:=\{a|k_a=k_{a-1}+1,\ 1\leqslant a\leqslant 2n\}. \tag{3.9}$$

Then the number of elements of A is equal to n, because we now set $k_{2n}=k_0=k$. We often use the abbreviations $(w)=\left(w_{a_1},\ldots,w_{a_n}\right)$, $(w')=\left(w_{a_1},\ldots,w_{a_{n-1}}\right)$ and $(w'')=\left(w_{a_1},\ldots,w_{a_{n-2}}\right)$ for $a_j\in A$ such that $a_1<\cdots<a_n$. Let us define the following meromorphic function:

$$Q^{(n)}(w|\zeta)^{kk_1\cdots k_{2n-1}k}=\prod_{a\in A}\left\{v_a-u_a+\frac{1}{2}-k_a\right\}\left(\prod_{j=1}^{a-1}\left[v_a-u_j-\frac{1}{2}\right]\prod_{j=a+1}^{2n}\left[u_j-v_a-\frac{1}{2}\right]\right)$$
$$\times\prod_{j=1}^{2n-1}\{k_j\}^{-1}\prod_{\substack{a,b\in A\\a<b}}[v_a-v_b+1]^{-1} \tag{3.10}$$

where $w_a=x^{2v_a}$. Here we should note that the structure of expression (3.10) is quite similar to that of (1.4).

We wish to find integral formulae of the form

$$\overline{F}_\sigma^{(n)}(\zeta)^{kk_1\cdots k_{2n-1}k}=\prod_{a\in A}\oint_{C_a}\frac{\mathrm{d}w_a}{2\pi\sqrt{-1}w_a}\Psi_\sigma^{(n)}(w|\zeta)Q^{(n)}(w|\zeta)^{kk_1\cdots k_{2n-1}k}. \tag{3.11}$$

Here, the kernel has the form

$$\Psi_\sigma^{(n)}(w|\zeta)=\vartheta_\sigma^{(n)}(w|\zeta)\prod_{a\in A}\prod_{j=1}^{2n}x^{-\frac{(v_a-u_j)^2}{2r}}\psi\left(\frac{w_a}{z_j}\right)\prod_{1\leqslant j<k\leqslant 2n}x^{-\frac{(u_j-u_k)^2}{4r}} \tag{3.12}$$

where

$$\psi(z)=\frac{(x^{2r+3}z;x^4,x^{2r})_\infty(x^{2r+3}z^{-1};x^4,x^{2r})_\infty}{(xz;x^4,x^{2r})_\infty(xz^{-1};x^4,x^{2r})_\infty}. \tag{3.13}$$

For the function $\vartheta_\sigma^{(n)}(w|\zeta)$ we assume that

- it is antisymmetric and holomorphic in the $w_a \in \mathbb{C}\backslash\{0\}$,
- it is symmetric and meromorphic in the $\zeta_j \in \mathbb{C}\backslash\{0\}$,
- it has the two transformation properties

$$\frac{\vartheta_\sigma^{(n)}(w|\zeta', x^2\zeta_{2n})}{\vartheta_\sigma^{(n)}(w|\zeta)} = \sigma x^{-2n+\frac{2n-1}{r}} \prod_{a\in A} w_a \prod_{j=1}^{2n} \zeta_j^{-1} \tag{3.14}$$

$$\frac{\vartheta_\sigma^{(n)}\left(w', x^4 w_{a_n}|\zeta\right)}{\vartheta_\sigma^{(n)}(w|\zeta)} = x^{-4n} \prod_{j=1}^{2n} \frac{z_j}{w_{a_n}}. \tag{3.15}$$

- it satisfies the following recursion relation:

$$\begin{aligned} \frac{\vartheta_\sigma^{(n)}(w', x^{-1}z_{2n-1}|\zeta'', \zeta_{2n-1}, \pm x^{-1}\zeta_{2n-1})}{\vartheta_{\pm\sigma}^{(n-1)}(w'|\zeta'')} &= \left(\pm x^{-1}\zeta_{2n-1}^2\right)^{-(n-\frac{1}{2})(1-\frac{1}{r})} \\ &\times \prod_{j=1}^{2n-2} \zeta_j^{-\frac{r-1}{r}} \prod_{\substack{a\in A\\ a\neq a_n}} \left(\pm x^{u_{2n-1}-v_a-\frac{1}{2}}\right)\Theta_{x^2}(xw_a/z_{2n-1}). \end{aligned} \tag{3.16}$$

Here we also fix the constant c_n as follows:

$$c_n = \frac{(-1)^{n(n+1)/2}x^{n^2/2}}{(x^2;x^2)_\infty^{n(n-3)/2}(x^{2r};x^{2r})_\infty^{3n(n-1)/2}} \frac{\{x^2\}_\infty^n\{x^6\}_\infty^n\{x^{2r+2}\}_\infty^n\{x^{2r+6}\}_\infty^n}{\{x^4\}_\infty^n\{x^8\}_\infty^n\{x^{2r}\}_\infty^n\{x^{2r+4}\}_\infty^n} \tag{3.17}$$

where

$$\{z\}_\infty := (z; x^4, x^4, x^{2r})_\infty.$$

The function $\vartheta_\sigma^{(n)}(w|\zeta)$ is otherwise arbitrary, and the choice of $\vartheta_\sigma^{(n)}(w|\zeta)$ corresponds to that of solutions to (3.2)–(3.5). The transformation property of $\vartheta_\sigma^{(n)}(w|\zeta)$ implies

$$\frac{\Psi_\sigma^{(n)}(w|\zeta', x^2\zeta_{2n})}{\Psi_\sigma^{(n)}(w|\zeta)} = \sigma \prod_{j=1}^{2n}\left(\frac{\zeta_{2n}}{\zeta_j}\right)^{\frac{r-1}{r}} \prod_{a\in A} \frac{\left[v_a - u_{2n} - \frac{1}{2}\right]}{\left[u_{2n} - v_a + \frac{3}{2}\right]} \tag{3.18}$$

$$\frac{\Psi_\sigma^{(n)}\left(w', x^4 w_{a_n}|\zeta\right)}{\Psi_\sigma^{(n)}(w|\zeta)} = \prod_{j=1}^{2n} \frac{\left[u_j - v_{a_n} - \frac{1}{2}\right]}{\left[v_{a_n} - u_j + \frac{3}{2}\right]}. \tag{3.19}$$

The integrand may have poles at

$$w_a = \begin{cases} x^{\pm(1+4k+2rl)}z_j & (1 \leqslant j \leqslant 2n, k, l \in \mathbb{Z}_{\geqslant 0}) \\ x^2 w_b & (b < a) \\ x^{-2}w_b & (b > a). \end{cases} \tag{3.20}$$

We choose the integration contour C_a with respect to w_a $(a \in A)$ such that C_a is along a simple closed curve oriented anticlockwise, and encircles the points $x^{1+4k+2rl}z_j$ $(1 \leqslant j \leqslant 2n, k, l \in \mathbb{Z}_{\geqslant 0})$ and $x^2 w_b$ $(b < a)$ but not $x^{-1-4k-2rl}z_j$ $(1 \leqslant j \leqslant 2n, k, l \in \mathbb{Z}_{\geqslant 0})$ nor $x^{-2}w_b$ $(b > a)$. Thus the contour C_a actually depends on z_j besides a, so that it should be denoted by $C_a(z) = C_a(z_1, \ldots, z_{2n})$, precisely. The LHS of (3.3) refers to the analytic continuation with respect to ζ_{2n}. Nevertheless, once we restrict ourselves to the principal regime $0 < p < x < \zeta_j^{-1} < 1$, we can tune all C_a to be the common integration contour $C : |w_a| = x^{-1}$ because of the inequality $xz_j < x^{-1} < x^{-1}z_j$.

We are now in a position to state the following proposition regarding the correlation functions and the bootstrap equations in the cyclic SOS model case.

Proposition 1. *Assume the properties of the function $\vartheta_\sigma^{(n)}$ below (3.13) and the integration contour C_a below (3.20). Then the integral formulae (3.6), (3.11) with (3.8), (3.10), (3.12), (3.13) solve the four equations (3.2)–(3.5).*

The proof of proposition 1 will be given in the subsequent subsections.

3.2. Proof of the W-symmetry

Let us first prove (3.2). For that purpose we have to consider four cases accordingly as $k_j - k_{j-1} = \pm 1$ and $k_{j+1} - k_j = \pm 1$.

Suppose that $k_j - k_{j-1} = k_{j+1} - k_j = -1$. Then relation (3.2) holds, because the integrand $\overline{F}_\sigma^{(n)}(\zeta)$ is evidently symmetric with respect to ζ_j and ζ_{j+1}.

Let $(k_{j-1}, k_j, k_{j+1}) = (m, m-1, m)$ for some m. Then relation (3.2) reduces to

$$\begin{aligned} \overline{F}_\sigma^{(n)}(\ldots,\zeta_{j+1},\zeta_j,\ldots)^{\cdots mm-1m\cdots} &= \frac{[1]\{m+u_j-u_{j+1}\}}{[u_j-u_{j+1}+1]\{m\}}\overline{F}_\sigma^{(n)}(\ldots,\zeta_j,\zeta_{j+1},\ldots)^{\cdots mm-1m\cdots} \\ &\quad - \frac{[u_j-u_{j+1}]\{m+1\}}{[u_j-u_{j+1}+1]\{m\}}\overline{F}_\sigma^{(n)}(\ldots,\zeta_j,\zeta_{j+1},\ldots)^{\cdots mm+1m\cdots}. \end{aligned} \tag{3.21}$$

Note that the set of integration variables in the second term of the RHS is different from the other terms. Since $w_a = x^{2v_a}$ are integration variables, we can replace both v_j and v_{j+1} in the integrand by v. After that, relation (3.21) follows from the equality of the integrands. In this step we use

$$\begin{aligned} \left\{v-u_j+\frac{1}{2}-m\right\}\left[v-u_{j+1}-\frac{1}{2}\right] &= \frac{[1]\{m+u_j-u_{j+1}\}}{[u_j-u_{j+1}+1]\{m\}}\left\{v-u_{j+1}+\frac{1}{2}-m\right\} \\ &\quad\times\left[v-u_j-\frac{1}{2}\right] - \frac{[u_j-u_{j+1}]\{m-1\}}{[u_j-u_{j+1}+1]\{m\}}\left\{v-u_j-\frac{1}{2}-m\right\}\left[u_{j+1}-v-\frac{1}{2}\right]. \end{aligned}$$

Suppose that $(k_{j-1}, k_j, k_{j+1}) = (m, m+1, m)$ for some m. This case can be proved in a similar way to the previous case. Here we use

$$\begin{aligned} \left\{v-u_{j+1}-\frac{1}{2}-m\right\}\left[u_j-v-\frac{1}{2}\right] &= \frac{[1]\{m-u_j+u_{j+1}\}}{[u_j-u_{j+1}+1]\{m\}}\left\{v-u_j-\frac{1}{2}-m\right\} \\ &\quad\times\left[u_{j+1}-v-\frac{1}{2}\right] - \frac{[u_j-u_{j+1}]\{m+1\}}{[u_j-u_{j+1}+1]\{m\}}\left\{v-u_{j+1}+\frac{1}{2}-m\right\}\left[v-u_j-\frac{1}{2}\right]. \end{aligned}$$

Finally, let $(k_{j-1}, k_j, k_{j+1}) = (m-1, m, m+1)$ for some m. Then the integrand on the RHS of (3.2) contains the factor

$$\begin{aligned} &I(u_j, u_{j+1}; v_j, v_{j+1}) \\ &\quad = \frac{\left\{v_j-u_j+\frac{1}{2}-m\right\}\left[u_{j+1}-v_j-\frac{1}{2}\right]\left\{v_{j+1}-u_{j+1}-\frac{1}{2}-m\right\}\left[v_{j+1}-u_j-\frac{1}{2}\right]}{[v_j-v_{j+1}+1]}. \end{aligned}$$

The corresponding factor on the LHS should be equal to $I(u_{j+1}, u_j; v_j, v_{j+1})$. Thus the difference between both sides contains the factor

$$I(u_j, u_{j+1}; v_j, v_{j+1}) - I(u_{j+1}, u_j; v_j, v_{j+1}) = \{m\}[u_j-u_{j+1}]\{v_j+v_{j+1}-u_j-u_{j+1}-m\}$$

which is symmetric with respect to $w_j = x^{2v_j}$ and $w_{j+1} = x^{2v_{j+1}}$. Since $\Psi_\sigma^{(n)}(w|\zeta)$ is antisymmetric with respect to w_a, relation (3.2) in this case does hold.

3.3. Proof of the cyclicity

In the proof of the cyclicity (3.3) we have to consider the two cases $k_{2n-1} = k \pm 1$. First let $k_{2n-1} = k + 1$. When integral (3.11) is analytically continued from $\zeta_{2n} = x^{u_{2n}}$ to $x^2\zeta_{2n} = x^{u_{2n}+2}$, the points $x^{1+4k+2rl}z_{2n}$ and $x^{-1-4k-2rl}z_{2n}$ ($k, l \in \mathbb{Z}_{\geqslant 0}$) move to the points $x^{5+4k+2rl}z_{2n}$ and $x^{3-4k-2rl}z_{2n}$, respectively. On the LHS of (3.3) the point $w_a = x^3 z_{2n}$ $\left(v_a = u_{2n} + \frac{3}{2}\right)$ is therefore outside the integral contour $C'_a = C_a(z', x^4 z_{2n})$. Nevertheless, we can deform C'_a to the original one $C_a = C_a(z)$ without crossing any poles. That is because the factor

$$\prod_{a\in A}\left[u_{2n} - v_a + \frac{3}{2}\right]$$

contained in $Q^{(n)}(w|\zeta', x^2\zeta_{2n})^{k\cdots k+1k}$ cancels the singularity at $w_a = x^3 z_{2n}$. Thus the integral contours for both sides of (3.3) coincide.

Furthermore, by using (3.18) we obtain

$$\Psi^{(n)}_\sigma(w|\zeta', x^2\zeta_{2n})\prod_{a\in A}\left[v_a - u_{2n} - \frac{3}{2}\right] = \sigma\Psi^{(n)}_\sigma(w|\zeta)\prod_{a\in A}\left[u_{2n} - v_a + \frac{1}{2}\right]\prod_{j=1}^{2n-1}\left(\frac{\zeta_j}{\zeta_{2n}}\right)^{\frac{1}{r}}$$

which implies that the integrands on both sides of (3.3) coincide and therefore relation (3.3) holds when $k_{2n-1} = k + 1$.

Next let $k_{2n-1} = k - 1$. In this case we rescale the variable $w_{2n} \mapsto x^4 w_{2n}$ ($v_{2n} \mapsto v_{2n} + 2$) on the LHS of (3.3). Then the integral contour with respect to w_a ($a \in A\backslash\{2n\}$) will be $C'_a = C_a(z', x^4 z_{2n})$, and the other one will be $\widetilde{C} = C_{2n}(x^{-4}z', z_{2n})$. For $a \in A\backslash\{2n\}$, we can deform the contour C'_a to the original C_a without crossing any poles, for the same reason as in the previous case. The integral contour $\widetilde{C}$ encircles $x^{-3+4k+2rl}z_j$ and $x^{1+4k+2rl}z_{2n}$, but not $x^{-5-4k-2rl}z_j$ nor $x^{-1-4k-2rl}z_{2n}$, where $1 \leqslant j \leqslant 2n-1$, $k, l \in \mathbb{Z}_{\geqslant 0}$. Since $Q^{(n)}(w|\zeta', x^2\zeta_{2n})^{k\cdots k-1k}$ contains the factor

$$J(w'x^4w_{2n}|\zeta', x^2\zeta_{2n}) = \left\{v_{2n} - u_{2n} + \frac{1}{2} - k\right\}\prod_{j=1}^{2n-1}\left[v_{2n} - u_j + \frac{3}{2}\right]\prod_{\substack{a\in A\\ a\neq 2n}}\frac{\left[u_{2n} - v_a + \frac{3}{2}\right]}{[v_a - v_{2n} - 1]} \tag{3.22}$$

the pole at $w_{2n} = x^{-3}z_j$ ($1 \leqslant j \leqslant 2n-1$) disappears. Thus we can deform the contour $\widetilde{C}$ to the original one $C_{2n} = C_{2n}(z)$ without crossing any poles. Thus the integral contours on both sides of (3.3) coincide.

Replace the integral variables such that $(w', w_{2n}) \mapsto (w_{2n}, w')$ on the RHS of (3.3), and compare the integrands on both sides. Note that (3.22) describes all w_{2n}- and z_{2n}-dependence of $Q^{(n)}(w|\zeta', x^2\zeta_{2n})^{k\cdots k-1k}$. From (3.18), (3.19) and the antisymmetric property of $\Psi^{(n)}_\sigma$ with respect to w_a, we have

$$\begin{aligned}\Psi^{(n)}_\sigma&(w', x^4w_{2n}|\zeta', x^2\zeta_{2n})J(w'x^4w_{2n}|\zeta', x^2\zeta_{2n})\\ &= \sigma\Psi^{(n)}_\sigma(w_{2n}, w'|\zeta)J(w_{2n}, w'|\zeta)\prod_{j=1}^{2n-1}\left(\frac{\zeta_j}{\zeta_{2n}}\right)^{\frac{1}{r}}\end{aligned} \tag{3.23}$$

which implies that the integrands on both sides of (3.3) coincide and therefore relation (3.3) holds when $k_{2n-1} = k - 1$.

3.4. Proof of the normalization condition

Let us prove (3.4) and (3.5). The factor $g(z_{2n-1}/z_{2n})$ has a zero at $\zeta_{2n} = \pm x^{-1}\zeta_{2n-1}$. On the other hand, the two points xz_{2n} and $x^{-1}z_{2n-1}$ are required to locate opposite sides of the integral contour C_a, so that a pinching may occur as $\zeta_{2n} \to \pm x^{-1}\zeta_{2n-1}$. If there is no pinching the correlation function $F_\sigma^{(n)}(\zeta)^{k\cdots k}$ vanishes at $\zeta_{2n} = \pm x^{-1}\zeta_{2n-1}$.

Suppose that $(k_{2n-2}, k_{2n-1}, k_{2n}) = (k+2, k+1, k)$. In this case the factor $\left[v_a - u_{2n-1} + \frac{1}{2}\right]$ contained in $Q^{(n)}(w|\zeta)^{k\cdots k+2k+1k}$ cancels the poles of $\psi(w_a/z_{2n-1})$ at $w_a = x^{-1}z_{2n-1}$, and therefore no pinching occurs as $\zeta_{2n} \to \pm x^{-1}\zeta_{2n-1}$. Thus conditions (3.4) and (3.5) hold when $(k_{2n-2}, k_{2n-1}, k_{2n}) = (k+2, k+1, k)$.

When $(k_{2n-2}, k_{2n-1}, k_{2n}) = (k-2, k-1, k)$, no pinching occurs for the integral contour for C_a $(a \in A\backslash\{2n-1, 2n\})$, for the same reason as in the previous case. Concerning the integral with respect to w_{2n-1} and w_{2n}, there are no singularities at $w_{2n-1} = xz_{2n}$ and $w_{2n} = xz_{2n}$ and consequently no pinching actually occurs. In order to see such vanishing singularities, we first note that the function $Q^{(n)}(w|\zeta)^{k\cdots k-2k-1k}$ contains the factor:

$$
\begin{aligned}
&q(v_{2n-1}, v_{2n}; u_{2n-1}, u_{2n}) \\
&= \frac{\left\{v_{2n-1} - u_{2n-1} + \frac{3}{2} - k\right\}\left[u_{2n} - v_{2n-1} - \frac{1}{2}\right]\left\{v_{2n} - u_{2n} + \frac{1}{2} - k\right\}\left[v_{2n} - u_{2n-1} - \frac{1}{2}\right]}{[v_{2n-1} - v_{2n} + 1]}.
\end{aligned}
$$

Because of the antisymmetric property of $\vartheta_\sigma^{(n)}$ with respect to w_a, the zero of $\vartheta_\sigma^{(n)}(w'', w_{2n-1}, xz_{2n}|\zeta)$ at $w_{2n-1} = xz_{2n}$ cancels the poles of $\psi(w_{2n-1}/z_{2n})$ at the same point. Furthermore, the poles of $\psi(w_{2n}/z_{2n})$ and $\psi(w_{2n}/z_{2n-1})$ at $w_{2n} = xz_{2n}$ and $z_{2n} = x^{-2}z_{2n-1}$ are cancelled by the zeros of $g(z_{2n-1}/z_{2n})$ and $Q^{(n)}(w'', w_{2n-1}, xz_{2n}|\zeta)^{k\cdots k-2k-1k}$ at the same points. The latter cancellation can be shown as follows. Note that the integral is invariant as we replace $Q^{(n)}(w'', w_{2n-1}, xz_{2n}|\zeta)^{k\cdots k-2k-1k}$ by its antisymmetric part with respect to w_{2n-1} and xz_{2n}. Accordingly, we can replace the factor $q\left(v_{2n-1}, u_{2n} + \frac{1}{2}; u_{2n-1}, u_{2n}\right)$ by the following factor:

$$
\begin{aligned}
&q\left(v_{2n-1}, u_{2n} + \tfrac{1}{2}; u_{2n-1}, u_{2n}\right) - q\left(u_{2n} + \tfrac{1}{2}, v_{2n-1}; u_{2n-1}, u_{2n}\right) \\
&\qquad = \left\{v_{2n-1} - u_{2n-1} + \tfrac{3}{2} - k\right\}\{1-k\}[u_{2n-1} - u_{2n}] \\
&\qquad\quad + \frac{\{u_{2n} - u_{2n-1} + 2 - k\}[1]\left\{v_{2n-1} - u_{2n} + \frac{1}{2} - k\right\}\left[v_{2n-1} - u_{2n-1} - \frac{1}{2}\right]}{\left[u_{2n} + \frac{3}{2} - v_{2n-1}\right]}
\end{aligned}
$$

that vanishes when $z_{2n} = x^{-2}z_{2n-1}$ $\left(u_{2n} = u_{2n-1} - 1 \text{ or } u_{2n} = u_{2n-1} - 1 - \frac{\pi\sqrt{-1}}{\epsilon}\right)$. Thus there is no singularity at $w_{2n} = xz_{2n}$ as $\zeta_{2n} \to \pm x^{-1}\zeta_{2n-1}$. The same thing at $w_{2n-1} = xz_{2n}$ can be easily shown in the same way. Hence conditions (3.4) and (3.5) are verified when $(k_{2n-2}, k_{2n-1}, k_{2n}) = (k-2, k-1, k)$.

Next let $(k_{2n-2}, k_{2n-1}, k_{2n}) = (k, k-1, k)$ and consider the limit $\zeta_{2n} \to x^{-1}\zeta_{2n-1}$. In this case there is no pinching for the integrals with respect to w_a $(a \in A\backslash\{2n\})$. Let $\widehat{C}$ denote the integral contour with respect to w_{2n} such that $\widehat{C}$ encircles the same points as C_{2n} does but xz_{2n}. Note that no pinching occurs with respect to the integral along $\widehat{C}$ because both the two points $x^{-1}z_{2n-1}$ and xz_{2n} lie outside the contour $\widehat{C}$. Thus the integral with respect to w_{2n} along the contour C_{2n} can be replaced by the residue at $w_{2n} = xz_{2n}$.

In order to evaluate the residue, the following formulae are useful:

$$
\lim_{z_{2n} \to x^{-2}z_{2n-1}} g\left(\frac{z_{2n-1}}{z_{2n}}\right)\psi\left(\frac{xz_{2n}}{z_{2n-1}}\right) = \frac{1}{(x^2; x^{2r})_\infty (x^{2r-2}; x^{2r})_\infty} \frac{\{x^4\}_\infty\{x^8\}_\infty\{x^{2r}\}_\infty\{x^{2r+4}\}_\infty}{\{x^2\}_\infty\{x^6\}_\infty\{x^{2r+2}\}_\infty\{x^{2r+6}\}_\infty} \tag{3.24}
$$

$$
\mathrm{Res}_{w_{2n}=xz_{2n}} \frac{\mathrm{d}w_{2n}}{w_{2n}}\psi\left(\frac{w_{2n}}{z_{2n}}\right) = \frac{1}{(x^2; x^2)_\infty (x^{2r}; x^{2r})_\infty} \tag{3.25}
$$

$$
g(z)g(x^2z)\psi(xz)[u+1] = x^{\frac{(u+1)^2}{r} - (u+1)}(x^{2r}; x^{2r})_\infty \tag{3.26}
$$

$$\psi\left(\frac{w}{z}\right)\psi\left(\frac{x^2w}{z}\right)\left[u-v-\frac{1}{2}\right]=\frac{x^{\frac{(u-v-1/2)^2}{r}-(u-v-1/2)}(x^{2r};x^{2r})_\infty}{(xw/z;x^2)_\infty(xz/w;x^2)_\infty}. \tag{3.27}$$

Using these, condition (3.4) with $(k_{2n-2},k_{2n-1},k_{2n})=(k,k-1,k)$ reduces to

$$\begin{aligned} c_{n-1}\vartheta_\sigma^{(n-1)}(w'|\zeta'') &= c_n\vartheta_\sigma^{(n)}(w',x^{-1}z_{2n-1}|\zeta'',\zeta_{2n-1},x^{-1}\zeta_{2n-1}) \\ &\times(-1)^n x^{-(2n-1)(1-\frac{1}{2r})}\zeta_{2n-1}^{(2n-1)(1-\frac{1}{r})}(x^2;x^2)_\infty^{n-2}(x^{2r};x^{2r})_\infty^{3(n-1)} \\ &\times\frac{\{x^4\}_\infty\{x^8\}_\infty\{x^{2r}\}_\infty\{x^{2r+4}\}_\infty}{\{x^2\}_\infty\{x^6\}_\infty\{x^{2r+2}\}_\infty\{x^{2r+6}\}_\infty}\prod_{j=1}^{2n-2}\zeta_j^{\frac{r-1}{r}}\prod_{\substack{a\in A\\ a\neq 2n}}\frac{x^{v_a-u_{2n-1}+\frac{1}{2}}}{\Theta_{x^2}(xw_a/z_{2n-1})}\end{aligned}$$

which is valid under the assumption of (3.16) and (3.17). Relation (3.5) with $(k_{2n-2},k_{2n-1},k_{2n})=(k,k-1,k)$ can be similarly proved.

When $(k_{2n-2},k_{2n-1},k_{2n})=(k,k+1,k)$, the only difference from the previous case is that the rational function $Q^{(n)}(w|\zeta)^{\cdots kk+1k}$ contains the factor

$$\left.\frac{\left\{v_{2n-1}-u_{2n-1}-\frac{1}{2}-k\right\}\left[u_{2n}-v_{2n-1}-\frac{1}{2}\right]}{\{k+1\}}\right|_{w_{2n-1}=xz_{2n}}=-\frac{\{u_{2n}-u_{2n-1}-k\}[1]}{\{k+1\}} \tag{3.28}$$

in the present case, while the corresponding factor in the previous case is

$$\left.\frac{\left\{v_{2n}-u_{2n}+\frac{1}{2}-k\right\}\left[v_{2n}-u_{2n-1}-\frac{1}{2}\right]}{\{k-1\}}\right|_{w_{2n}=xz_{2n}}=[u_{2n}-u_{2n-1}]. \tag{3.29}$$

Since (3.28) is equal to (3.29) as $\zeta_{2n}=x^{-1}\zeta_{2n-1}$, condition (3.4) with $(k_{2n-2},k_{2n-1},k_{2n})=(k,k+1,k)$ follows from the previous case. Condition (3.5) with $(k_{2n-2},k_{2n-1},k_{2n})=(k,k+1,k)$ can be similarly proved. Finally, conditions (3.4) and (3.5) were proved componentwise.

3.5. Non-trivial theta function

In subsections 3.2–3.4 we proved proposition 1. The key point in proving (3.2) was the W-matrix symmetry of the rational function $Q^{(n)}(w|\zeta)^{k\cdots k}$. We observed that the other two conditions (3.3) and (3.4) hold under the assumption of the transformation properties and the recursion relation of $\vartheta_\sigma^{(n)}$. Actually, you can easily find that (3.14), (3.15) and (3.16) are sufficient conditions of (3.3) for $k_{2n-1}=k+1$, (3.3) for $k_{2n-1}=k-1$ and (3.4), respectively.

In this way we obtained the integral solutions to CTM bootstrap equations for correlation functions of the cyclic SOS model, with the freedom of the choice of $\vartheta_\sigma^{(n)}$. Now we wish to present an example of $\vartheta_\sigma^{(n)}$ satisfying all the properties given below (3.13).

$$\vartheta_\sigma^{(n)}(w|\zeta)=\Theta_{x^2}\left(-\sigma x\prod_{a\in A}w_a^{-1}\prod_{j=1}^{2n}\zeta_j\right)\prod_{j=1}^{2n}\zeta_j^{-(n-\frac{1}{2})(1-\frac{1}{r})}\prod_{\substack{a,b\in A\\ a<b}}x^{v_b-v_a}\Theta_{x^2}(w_a/w_b). \tag{3.30}$$

You can easily see that (3.30) satisfies all the properties of symmetry with respect to ζ_j, antisymmetry with respect to w_a, (3.14), (3.15) and (3.16). We also note that there are actually no poles at $w_a=x^2w_b$ $(b<a)$ and $w_a=x^{-2}w_b$ $(b>a)$ when we fix $\vartheta_\sigma^{(n)}$ to (3.30) because of the factor $\Theta_{x^2}(w_a/w_b)$. Furthermore, we note that this example of the function $\vartheta_\sigma^{(n)}(w|\zeta)$ is quite similar to the antiferromagnetic XXZ analogue [20].

4. Integral formulae for the XYZ antiferromagnet

4.1. Correlation functions and difference equations

Let us introduce the $V^{\otimes 2n}$-valued correlation functions

$$G_\sigma^{(n)}(\zeta_1,\ldots,\zeta_{2n}) = \sum_{\varepsilon_j=\pm} v_{\varepsilon_1}\otimes\ldots\otimes v_{\varepsilon_{2n}} G_\sigma^{(n)}(\zeta_1,\ldots,\zeta_{2n})^{\varepsilon_1\cdots\varepsilon_{2n}} \tag{4.1}$$

where $\sigma=\pm$ signifies one of the two vacuums of the XYZ Heisenberg antiferromagnet. Let

$$\mathcal{O} = E^{(1)}_{\varepsilon_1\varepsilon'_1}\cdots E^{(n)}_{\varepsilon_n\varepsilon'_n} \tag{4.2}$$

where $E^{(j)}_{\varepsilon_j\varepsilon'_j}$ is the matrix unit on the jth site. Then the correlation function (4.1) gives the expectation value of the local operator (4.2) by specializing the spectral parameters as follows:

$$\langle\mathcal{O}\rangle_\sigma = G_\sigma^{(n)}(\overbrace{x^{-1}\zeta,\ldots,x^{-1}\zeta}^{n},\overbrace{\zeta,\ldots,\zeta}^{n})^{-\varepsilon_n\cdots-\varepsilon_1\varepsilon'_1\cdots\varepsilon'_n}. \tag{4.3}$$

In what follows we often use the abbreviations: $(\zeta)=(\zeta_1,\ldots,\zeta_{2n})$, $(\zeta')=(\zeta_1,\ldots,\zeta_{2n-1})$, $(\zeta'')=(\zeta_1,\ldots,\zeta_{2n-2})$, $(z)=(z_1,\ldots,z_{2n})$, $(z')=(z_1,\ldots,z_{2n-1})$; and $(\varepsilon)=(\varepsilon_1\cdots\varepsilon_{2n})$, $(\varepsilon')=(\varepsilon_1\cdots\varepsilon_{2n-1})$, $(\varepsilon'')=(\varepsilon_1\cdots\varepsilon_{2n-2})$. On the basis of the CTM (corner transfer matrix) bootstrap approach, the correlation functions satisfy the following three conditions [7]:

1. *R-matrix symmetry*

$$P_{jj+1}G_\sigma^{(n)}(\ldots,\zeta_{j+1},\zeta_j,\ldots) = R_{jj+1}(\zeta_j/\zeta_{j+1})G_\sigma^{(n)}(\ldots,\zeta_j,\zeta_{j+1},\ldots) \qquad (1\leqslant j\leqslant 2n-1) \tag{4.4}$$

where $P(x\otimes y)=y\otimes x$.

2. *Cyclicity*

$$P_{12}\cdots P_{2n-12n}G_\sigma^{(n)}(\zeta',x^2\zeta_{2n}) = \sigma G_\sigma^{(n)}(\zeta_{2n},\zeta'). \tag{4.5}$$

3. *Normalization*

$$G_\sigma^{(n)}(\zeta'',\zeta_{2n-1},\zeta_{2n})|_{\zeta_{2n}=sx^{-1}\zeta_{2n-1}} = G_{s\sigma}^{(n-1)}(\zeta'')\otimes u_s \qquad (s=\pm) \tag{4.6}$$

where $u_s = v_+\otimes v_- + sv_-\otimes v_+$.

These three conditions can be recast componentwise as follows:

$$G_\sigma^{(n)}(\ldots,\zeta_{j+1},\zeta_j,\ldots)^{\cdots\varepsilon_{j+1}\varepsilon_j\cdots} = \sum_{\varepsilon'_j,\varepsilon'_{j+1}=\pm} R(\zeta_j/\zeta_{j+1})^{\varepsilon_j\varepsilon_{j+1}}_{\varepsilon'_j\varepsilon'_{j+1}} G_\sigma^{(n)}(\ldots,\zeta_j,\zeta_{j+1},\ldots)^{\cdots\varepsilon'_j\varepsilon'_{j+1}\cdots} \tag{4.7}$$

$$G_\sigma^{(n)}(\zeta',x^2\zeta_{2n})^{\varepsilon'\varepsilon_{2n}} = \sigma G_\sigma^{(n)}(\zeta_{2n},\zeta')^{\varepsilon_{2n}\varepsilon'} \tag{4.8}$$

$$G_\sigma^{(n)}(\zeta'',\zeta_{2n-1},sx^{-1}\zeta_{2n-1})^{\varepsilon''\varepsilon_{2n-1}\varepsilon_{2n}} = \varepsilon_{2n-1}^{(1-s)/2}\delta_{\varepsilon_{2n-1}+\varepsilon_{2n},0}G_{s\sigma}^{(n-1)}(\zeta'')^{\varepsilon''} \qquad (s=\pm). \tag{4.9}$$

Combining (4.7) and (4.9) we obtain another expression for the normalization condition:

$$\sum_{s=\pm}\varepsilon^{(1-s)/2}G_\sigma^{(n)}(\zeta'',\zeta_{2n-1},sx\zeta_{2n-1})^{\varepsilon''\varepsilon-\varepsilon} = G_{s\sigma}^{(n-1)}(\zeta'')^{\varepsilon''}$$

where we also use

$$R(sx^{-1}) = \begin{pmatrix} 0&0&0&0\\ 0&s&1&0\\ 0&1&s&0\\ 0&0&0&0\end{pmatrix}.$$

Note that the first two conditions imply the difference equation of the quantum KZ type [31] of level -4:

$$T_j G^{(n)}_\sigma(\zeta) = R_{jj-1}(x^{-2}\zeta_j/\zeta_{j-1})\cdots R_{j1}(x^{-2}\zeta_j/\zeta_1)R_{j2n}(\zeta_j/\zeta_{2n})\cdots R_{jj+1}(\zeta_j/\zeta_{j+1})G^{(n)}_\sigma(\zeta) \tag{4.10}$$

where T_j is the shift operator such that

$$T_j F(\zeta) = F(\zeta_1,\ldots,x^{-2}\zeta_j,\ldots,\zeta_{2n}) \tag{4.11}$$

for any $2n$-variable function F. When the definition of T_j is replaced by

$$T_j F(\zeta) = F(\zeta_1,\ldots,x^{l+2}\zeta_j,\ldots,\zeta_{2n})$$

and all the arguments $x^{-2}\zeta_j/\zeta_k$ in the first line of the RHS of (4.10) are also replaced by $x^{l+2}\zeta_j/\zeta_k$, the difference equation (4.10) is called the quantum KZ equations of level l.

Remark. As a result of $\mathbb{Z}_2$-symmetry of R-matrix, there are two ground states in the XYZ antiferromagnet. Let us specify the two ground states by $i = 0, 1$, and denote the correlation function on the ith ground state by $G^{(n)}_i(\zeta)$. Theoretically speaking, representation of the correlation function $G^{(n)}_i(\zeta)$ refers to the trace of type I vertex operators on the irreducible highest weight module $\mathcal{H}^{(i)}$ of $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}_2})$ [26]. The CTM bootstrap approach suggests that both $G^{(n)}_0(\zeta)$ and $G^{(n)}_1(\zeta)$ will appear in the cyclicity condition as follows:

$$P_{12}\cdots P_{2n-12n}G^{(n)}_i(\zeta_1,\ldots,\zeta_{2n-1},x^2\zeta_{2n}) = G^{(n)}_{1-i}(\zeta_{2n},\zeta_1,\ldots,\zeta_{2n-1}).$$

Thus we introduce $G^{(n)}_\sigma(\zeta) = G^{(n)}_0(\zeta) + \sigma G^{(n)}_1(\zeta)$ such that the second equation (4.5) involves only $G^{(n)}_\sigma(\zeta)$.

Thanks to (3.7) the first two equations (4.7) and (4.8) are rephrased in terms of $\overline{G}^{(n)}_\sigma(\zeta)$ and $\overline{R}(\zeta)$ as follows:

$$\overline{G}^{(n)}_\sigma(\ldots,\zeta_{j+1},\zeta_j,\ldots)^{\cdots\varepsilon_{j+1}\varepsilon_j\cdots} = \sum_{\varepsilon'_j,\varepsilon'_{j+1}=\pm}\overline{R}(\zeta_j/\zeta_{j+1})^{\varepsilon_j,\varepsilon_{j+1}}_{\varepsilon'_j,\varepsilon'_{j+1}}\overline{G}^{(n)}_\sigma(\ldots,\zeta_j,\zeta_{j+1},\ldots)^{\varepsilon'_j,\varepsilon'_{j+1}} \tag{4.12}$$

$$\overline{G}^{(n)}_\sigma(\zeta',x^2\zeta_{2n})^{\varepsilon'\varepsilon_{2n}} = \sigma\overline{G}^{(n)}_\sigma(\zeta_{2n},\zeta')^{\varepsilon_{2n}\varepsilon'}\prod_{j=1}^{2n-1}\left(\frac{\zeta_{2n}}{\zeta_j}\right)^{\frac{r-1}{r}}. \tag{4.13}$$

4.2. Generalized correlation functions of the SOS model

For later convenience we also introduce another meromorphic function $Q^{(n)}(w|\zeta)^{kk_1\cdots k_{2n-1}k-2l}$, where $0 \leqslant l \leqslant n$. Since $k_{2n} = k - 2l$, the number of elements of the following set A' is equal to n:

$$A' = A \sqcup \{2n+1,\ldots,2n+l\}.$$

Let $k_{2n+i} = k_{2n} + 2i = k - 2(l-i)$ for $1 \leqslant i \leqslant l$. Then the meromorphic function $Q^{(n)}(w|\zeta)^{kk_1\cdots k_{2n-1}k-2l}$ is defined as follows:

$$\begin{aligned}Q^{(n)}(w|\zeta)^{kk_1\cdots k_{2n-1}k-2l} &= \prod_{a\in A}\left\{v_a - u_a + \frac{1}{2} - k_a\right\}\left(\prod_{j=1}^{a-1}\left[v_a - u_j - \frac{1}{2}\right]\prod_{j=a+1}^{2n}\left[u_j - v_a - \frac{1}{2}\right]\right)\\ &\quad\times(-1)^l\prod_{a'=2n+1}^{2n+l}\frac{\left\{v_{a'} - u_0 + \frac{1}{2} - k_{a'}\right\}}{\left[v_{a'} - u_0 - \frac{1}{2}\right]}\left(\prod_{j=1}^{2n}\left[v_{a'} - u_j - \frac{1}{2}\right]\right)\\ &\quad\times\prod_{j=1}^{2n-1}\{k_j\}^{-1}\prod_{\substack{a,b\in A'\\ a<b}}[v_a - v_b + 1]^{-1}.\end{aligned} \tag{4.14}$$

When $l = 0$, the meromorphic function (4.14) is evidently reduced to the expression of (3.10).

We further introduce the generalized correlation function of the cyclic SOS model as follows:

$$\overline{F}_\sigma^{(n)}(\zeta)^{kk_1\cdots k_{2n-1}k-2l} = \prod_{a\in A'} \oint_{C_a} \frac{\mathrm{d}w_a}{2\pi\sqrt{-1}w_a} \Psi_\sigma^{(n)}(w|\zeta) Q^{(n)}(w|\zeta)^{kk_1\cdots k_{2n-1}k-2l} \tag{4.15}$$

where the integral kernel $\Psi_\sigma^{(n)}(w|\zeta)$ and the integral contour C_a are, respectively, the same as in the previous section.

Formula (4.15) can be used only for $k_{2n} \leqslant k$. However, by noticing the fact that the Boltzmann weights of the cyclic SOS model (2.10) are invariant under the shift $(a, b, c, d) \mapsto (-a, -b, -c, -d)$, we should obtain the expression for $k_{2n} = k + 2l > k$ from $\overline{F}_\sigma^{(n)}(\zeta)^{-k-k_1\cdots-k_{2n-1}-k-2l}$ as follows:

$$\overline{F}_\sigma^{(n)}(\zeta)^{kk_1\cdots k_{2n-1}k+2l} = \prod_{a\in A'_-} \oint_{C_a} \frac{\mathrm{d}w_a}{2\pi\sqrt{-1}w_a} \Psi_\sigma^{(n)}(w|\zeta) Q^{(n)}(w|\zeta)^{kk_1\cdots k_{2n-1}k+2l} \tag{4.16}$$

where

$$\begin{aligned} Q^{(n)}(w|\zeta)^{kk_1\cdots k_{2n-1}k+2l} &= \prod_{a\in A_-} \left\{ v_a - u_a + \frac{1}{2} + k_a \right\} \left(\prod_{j=1}^{a-1} \left[v_a - u_j - \frac{1}{2} \right] \prod_{j=a+1}^{2n} \left[u_j - v_a - \frac{1}{2} \right] \right) \\ &\quad \times (-1)^l \prod_{a'=2n+1}^{2n+l} \frac{\left\{ v_{a'} - u_0 + \frac{1}{2} + k_{a'} \right\}}{\left[v_{a'} - u_0 - \frac{1}{2} \right]} \left(\prod_{j=1}^{2n} \left[v_{a'} - u_j - \frac{1}{2} \right] \right) \\ &\quad \times \prod_{j=1}^{2n-1} \{k_j\}^{-1} \prod_{\substack{a,b\in A'_- \\ a<b}} [v_a - v_b + 1]^{-1}. \end{aligned} \tag{4.17}$$

Here

$$A_- := \{a | k_a = k_{a-1} - 1, 1 \leqslant a \leqslant 2n\} \qquad A'_- = A_- \sqcup \{2n+1, \ldots, 2n+l\}$$

and $k_{2n+i} = k_{2n} - 2i = k + 2(l - i)$ for $1 \leqslant i \leqslant l$.

4.3. First integral formula for the XYZ correlation functions

Let

$$\begin{aligned} G_\sigma^{(n)}(\zeta_1, \ldots, \zeta_{2n}) &= (\sqrt{-1})^{-n} \sum_{k_0, k_1, \ldots, k_{2n}} t_{k_1}^{k_0}(u_1 - u_0) \otimes \cdots \otimes t_{k_{2n}}^{k_{2n-1}}(u_{2n} - u_0) \\ &\quad \times F_\sigma^{(n)}(\zeta_1, \ldots, \zeta_{2n})^{k_0k_1\cdots k_{2n}}. \end{aligned} \tag{4.18}$$

Here $\zeta_j = x^{u_j}$, and

$$F_\sigma^{(n)}(\zeta)^{kk_1\cdots k_{2n}} = c_n \prod_{1\leqslant j<k\leqslant 2n} \zeta_j^{\frac{r-1}{r}} g(z_j/z_k) \times \overline{F}_\sigma^{(n)}(\zeta)^{kk_1\cdots k_{2n}}$$

where $\overline{F}_\sigma^{(n)}(\zeta)^{kk_1\cdots k_{2n}}$ is defined by (4.15), (4.14) for $k_{2n} \leqslant k$, otherwise by (4.16), (4.17). The sum with respect to k_j $(1 \leqslant j \leqslant 2n)$ should be taken over $k_j = k_{j-1} \pm 1$. The sum with respect to k_0 should be as follows. When $r \in \mathbb{Q}$ and $2r = N/N'$ (N, N' are coprime), we have

$$t_{N+k\pm1}^{N+k}(u) = (-1)^{N'} t_{k\pm1}^{k}(u).$$

The cyclic SOS correlation function $F_\sigma^{(n)}(\zeta)^{kk_1\cdots k_{2n}}$ is evidently invariant under the shift $(k, k_1, \ldots, k_{2n}) \mapsto (k+N, k_1+N, \ldots, k_{2n}+N)$. Thus the summand in (4.18) is also invariant under the same shift, so that the sum with respect to k_0 can be taken over $\mathbb{Z}/N\mathbb{Z}$. Since there is no such invariance, the sum with respect to k_0 should be taken over $\mathbb{Z}$ when r is irrational. From these observations we have

$$\sum_{k_0} = \begin{cases} \sum_{k_0=0}^{N-1} & \text{if} \quad r \in \mathbb{Q} \quad 2r = N/N' \quad (N, N' \text{ are coprime}) \\ \sum_{k_0\in\mathbb{Z}} & \text{if} \quad r \notin \mathbb{Q}. \end{cases} \tag{4.19}$$

Simple observation shows that the R-matrix symmetry (4.4) follows from the W-symmetry (3.2) for (4.15), (4.16). The cyclicity for the generalized correlation functions in the cyclic SOS model does not hold but we have the following relations.

Proposition 2. *For fixed* $k_0 = k, k_1, \ldots, k_{2n-2}, k_{2n-1} = k'$ *with* $k_j = k_{j-1} \pm 1$ $(1 \leqslant j \leqslant 2n-1)$*, the following cyclicity relations hold:*

$$\sum_{s=\pm1} t_{k'+s}^{k'}(u_{2n} - u_0 + 2) F_\sigma^{(n)}(\zeta_1, \ldots, \zeta_{2n-1}, x^2\zeta_{2n})^{kk_1\cdots k_{2n-2}k'k'+s}$$
$$= \sum_{s=\pm1} t_k^{k-s}(u_{2n} - u_0) F_\sigma^{(n)}(\zeta_{2n}, \zeta_1, \ldots, \zeta_{2n-1})^{k-skk_1\cdots k_{2n-2}k'}. \tag{4.20}$$

Proof. Let $k' = k - 2l + 1$ with $1 \leqslant l \leqslant n$. (The case $k' = k + 2l - 1$ can be similarly proved.) For $\varepsilon = \pm$ let

$$h_\varepsilon(u) := \theta_{2-\varepsilon}\theta_{3+\varepsilon}\left(\frac{u}{2r}; \frac{\pi\sqrt{-1}}{\epsilon r}\right) = \begin{cases} \theta_1\theta_4\left(\frac{u}{2r}; \frac{\pi\sqrt{-1}}{\epsilon r}\right) & (\varepsilon > 0) \\ \theta_2\theta_3\left(\frac{u}{2r}; \frac{\pi\sqrt{-1}}{\epsilon r}\right) & (\varepsilon < 0). \end{cases}$$

Then from (2.7), (2.9) relation (4.20) is reduced to

$$(-1)^l[u_{2n} - u_0 + 1] \sum_{s=\pm1} h_\varepsilon(k' - s(u_{2n} - u_0 + 2)) F_\sigma^{(n)}(\zeta', x^2\zeta_{2n})^{kk_1\cdots k_{2n-2}k'k'+s}$$
$$+ [u_{2n} - u_0 + 2] \sum_{s=\pm1} s h_\varepsilon(k - s(u_{2n} - u_0 + 1)) F_\sigma^{(n)}(\zeta_{2n}, \zeta')^{k-skk_1\cdots k_{2n-2}k'} = 0. \tag{4.21}$$

Here we used the abbreviation $\zeta' = (\zeta_1, \ldots, \zeta_{2n-1})$ and the functional relation

$$\frac{f(u+2)}{f(u)} = \frac{[u+1]}{[u+2]}.$$

Simple calculations show that

$$\sum_{s=\pm1} h_\varepsilon(k' - s(u_{2n} - u_0)) F_\sigma^{(n)}(\zeta', \zeta_{2n})^{kk_1\cdots k_{2n-2}k'k'+s}$$
$$= \prod_{a\in A'} \oint_{C_a} \frac{\mathrm{d}w_a}{2\pi\sqrt{-1}w_a} \Psi_\sigma^{(n)}(w_{a_1}, \ldots, w_{2n+l}|\zeta_1, \ldots, \zeta_{2n}) \mathcal{F}_{\varepsilon,l}(v|u_0, u_1, \ldots, u_{2n}) \tag{4.22}$$

where

$$\begin{aligned}\mathcal{F}_{\varepsilon,l}(v|u_0,u_1,\ldots,u_{2n}) &= h_\varepsilon(k-2l+2+u_0+u_{2n}-2v_{2n+1})\\ &\times\frac{[u_{2n}-u_0]}{\left[v_{2n+1}-u_0-\frac{1}{2}\right]}\prod_{i=2}^{l}\frac{\left\{v_{2n+i}-u_0+\frac{1}{2}-k+2(l-i)\right\}}{\left[v_{2n+i}-u_0-\frac{1}{2}\right]}\left[v_{2n+i}-u_{2n}-\frac{1}{2}\right]\\ &\times\prod_{\substack{a\in A\\ a<2n}}\left\{v_a-u_a+\frac{1}{2}-k_a\right\}\left(\prod_{j=1}^{a-1}\left[v_a-u_j-\frac{1}{2}\right]\prod_{j=a+1}^{2n}\left[u_j-v_a-\frac{1}{2}\right]\right)\\ &\times(-1)^{l-1}\prod_{a'=2n+1}^{2n+l}\prod_{j=1}^{2n-1}\left[v_{a'}-u_j-\frac{1}{2}\right]\prod_{j=1}^{2n-2}\{k_j\}^{-1}\prod_{\substack{a,b\in A'\\ a<b}}[v_a-v_b+1]^{-1}.\end{aligned}\tag{4.23}$$

It follows from the definition that $2n\in A$ and $A'=A\sqcup\{2n+1,\ldots,2n+l-1\}$ when $s=1$, and that $2n\notin A$ and $A'=A\sqcup\{2n+1,\ldots,2n+l\}$ when $s=-1$ on the LHS of (4.22). Thus we made the shift of variables $(w_{2n},\ldots,w_{2n+l-1})\mapsto(w_{2n+1},\ldots,w_{2n+l})$ for $s=1$ in (4.22). Furthermore, in order to derive the expression of (4.23) we used the identity

$$\begin{aligned}&\frac{\left[v-u-\frac{1}{2}\right]}{\left[v-u_0-\frac{1}{2}\right]}\left\{v-u_0-\frac{1}{2}-k\right\}h_\varepsilon(k+u-u_0)-\left\{v-u-\frac{1}{2}-k\right\}h_\varepsilon(k-u+u_0)\\ &\qquad=-\frac{[u-u_0]\{k\}h_\varepsilon(k+1+u_0+u-2v)}{\left[v-u_0-\frac{1}{2}\right]}.\end{aligned}$$

By repeating similar calculations we find that equation (4.21) is reduced to

$$\prod_{a\in A'}\oint_{C_a}\frac{\mathrm{d}w_a}{2\pi\sqrt{-1}w_a}\Psi_\sigma^{(n)}(w|\zeta_1,\ldots,\zeta_{2n})\mathcal{A}_l(v|u)\mathcal{B}_{\varepsilon,k,l}(v_{2n+1},\ldots,v_{2n+l}|u_0,u_{2n})=0\tag{4.24}$$

where

$$\begin{aligned}\mathcal{A}_l &= \prod_{\substack{a\in A\\ a<2n}}\left\{v_a-u_a+\frac{1}{2}-k_a\right\}\left(\left[v_a-u_{2n}-\frac{1}{2}\right]\prod_{j=1}^{a-1}\left[v_a-u_j-\frac{1}{2}\right]\prod_{j=a+1}^{2n-1}\left[u_j-v_a-\frac{1}{2}\right]\right)\\ &\times\prod_{i=1}^{2n+l}\left[v_{2n+i}-u_0+\frac{3}{2}\right]^{-1}\left(\prod_{j=1}^{2n-1}\left[u_j-v_{2n+i}-\frac{1}{2}\right]\right)\prod_{j=1}^{2n-2}\{k_j\}^{-1}\\ &\times\prod_{\substack{a,b\in A'\\ a<b<2n}}[v_a-v_b+1]^{-1}\prod_{\substack{a,a'\in A'\\ a<2n<a'}}[v_a-v_{a'}-1]^{-1}\prod_{\substack{a',b'\in A'\\ 2n<a'<b'}}[v_{a'}-v_{b'}+1]^{-1}\end{aligned}$$

$$\begin{aligned}\mathcal{B}_{\varepsilon,k,l} &= h_\varepsilon(k-2l+u_0+u_{2n}-2v_{2n+1})\prod_{i=2}^{l}\left\{v_{2n+i}-u_0+\frac{5}{2}-k+2(l-i)\right\}\\ &\times\left[v_{2n+i}-u_{2n}-\frac{1}{2}\right]+(-1)^l h_\varepsilon(k-2+u_0+u_{2n}-2v_{2n+l})\\ &\times\prod_{i=1}^{l-1}\left\{v_{2n+i}-u_0+\frac{3}{2}-k+2(l-i)\right\}\left[u_{2n}-v_{2n+i}-\frac{1}{2}\right].\end{aligned}$$

It is evident that (4.24) for $l = 1$ holds because $\mathcal{B}_{\varepsilon,k,1} = 0$. In order to prove (4.24) for $l > 1$, we wish to show that

$$\mathcal{B}_{\varepsilon,k,l}(v|u_0, u_{2n}) = \sum_{i=1}^{l-1}[v_{2n+i} - v_{2n+i+1} + 1]B^{(i)}_{\varepsilon,k,l}(v|u_0, u_{2n}) \tag{4.25}$$

where $B^{(i)}_{\varepsilon,k,l}$ is some function that is symmetric with respect to v_{2n+i} and v_{2n+i+1}. Suppose that (4.25) is true. Then $\mathcal{A}_l\mathcal{B}_{\varepsilon,k,l}$ is the sum of $l-1$ terms, each of which is symmetric with respect to v_{2n+i} and v_{2n+i+1}. Correspondingly, integral (4.24) vanishes from the antisymmetry of $\Psi^{(n)}_\sigma$ with respect to w. The claim of this proposition is, therefore, reduced to equation (4.25).

Let us show (4.25) by induction for $l > 1$. The validity of equation (4.25) for $l = 2$ follows from the following identity:

$$\begin{aligned}
&\mathcal{B}_{\varepsilon,k,2}(v_{2n+1}, v_{2n+2}|u_0, u_{2n}) \\
&\quad = h_\varepsilon(k - 4 + u_0 + u_{2n} - 2v_{2n+1})\left\{v_{2n+2} - u_0 + \tfrac{5}{2} - k\right\}\left[v_{2n+2} - u_{2n} - \tfrac{1}{2}\right] \\
&\qquad + h_\varepsilon(k - 2 + u_0 + u_{2n} - 2v_{2n+2})\left\{v_{2n+1} - u_0 + \tfrac{7}{2} - k\right\}\left[u_{2n} - v_{2n+1} - \tfrac{1}{2}\right] \\
&\quad = -h_\varepsilon(k - 3 + u_0 - u_{2n})\{k - 3 + u_0 + u_{2n} - v_{2n+1} - v_{2n+2}\}[v_{2n+1} - v_{2n+2} + 1].
\end{aligned} \tag{4.26}$$

For $l > 2$, the assumption of the induction implies

$$\begin{aligned}
&\mathcal{B}_{\varepsilon,k-2,l-1}(v_{2n+1}, \ldots, v_{2n+l-1}|u_0, u_{2n}) \\
&\quad = \sum_{i=1}^{l-2}[v_{2n+i} - v_{2n+i+1} + 1]B^{(i)}_{\varepsilon,k-2,l-1}(v_{2n+1}, \ldots, v_{2n+l-1}|u_0, u_{2n}).
\end{aligned}$$

The validity of equation (4.25) for general l thus follows from the relation:

$$\begin{aligned}
&\mathcal{B}_{\varepsilon,k,l}(v_{2n+1}, \ldots, v_{2n+l}|u_0, u_{2n}) \\
&\quad = \mathcal{B}_{\varepsilon,k-2,l-1}(v_{2n+1}, \ldots, v_{2n+l-1}|u_0, u_{2n})\left\{v_{2n+l} - u_0 + \frac{5}{2} - k\right\} \\
&\qquad \times \left[v_{2n+l} - u_{2n} - \frac{1}{2}\right] + (-1)^l\mathcal{B}_{\varepsilon,k,2}(v_{2n+l-1}, v_{2n+l}|u_0, u_{2n}) \\
&\qquad \times \prod_{i=1}^{l-2}\left\{v_{2n+i} - u_0 + \frac{3}{2} - k + 2(l-i)\right\}\left[u_{2n} - v_{2n+i} - \frac{1}{2}\right]
\end{aligned}$$

where we again use (4.26). □

As a corollary of proposition 2, we have

Corollary 3. *The cyclicity (4.5) holds for the XYZ correlation functions (4.18).*

4.4. Normalization in the XYZ case

Furthermore, we find that the following normalization condition holds for the generalized correlation functions in the cyclic SOS model for $k_{2n} \equiv k \pmod 2$:

$$F^{(n)}_\sigma(\zeta'', \zeta_{2n-1}, sx^{-1}\zeta_{2n-1})^{k\cdots k'k'\pm 1k_{2n}} = \delta_{k',k_{2n}}\frac{d_s}{\{k_{2n}\}}F^{(n-1)}_\sigma(\zeta'')^{k\cdots k'} \qquad (s = \pm) \tag{4.27}$$

where

$$d_s = \begin{cases} 1 & (s > 0) \\ \mathrm{e}^{-\frac{\pi\sqrt{-1}}{2r}(1+r\pm 2k)} & (s < 0). \end{cases}$$

Proposition 4. *The normalization condition (4.6) holds for the XYZ correlation functions (4.18).*

Proof. First we note the following identity:

$$\theta_1\theta_4\left(\frac{k+u}{2r};\frac{\pi\sqrt{-1}}{\epsilon r}\right)\theta_2\theta_3\left(\frac{k-u}{2r};\frac{\pi\sqrt{-1}}{\epsilon r}\right)$$
$$-\theta_1\theta_4\left(\frac{k-u}{2r};\frac{\pi\sqrt{-1}}{\epsilon r}\right)\theta_2\theta_3\left(\frac{k+u}{2r};\frac{\pi\sqrt{-1}}{\epsilon r}\right)$$
$$=\theta_2\theta_3\left(0;\frac{\pi\sqrt{-1}}{\epsilon r}\right)\theta_1\left(\frac{u}{2r};\frac{\pi\sqrt{-1}}{\epsilon r}\right)\theta_4\left(\frac{k}{2r};\frac{\pi\sqrt{-1}}{\epsilon r}\right)=C^2[u]\{k\} \tag{4.28}$$

where C is the same constant as the one defined in (2.8). Using (4.28) and (2.8), we have

$$\sum_{s=\pm1} t_{k+s}^{k}(u_{2n-1}-u_0)^{\varepsilon_{2n-1}} t_k^{k+s}(u_{2n}-u_0)^{\varepsilon_{2n}}\Big|_{u_{2n}=u_{2n-1}-1} = \sqrt{-1}\{k\}\delta_{\varepsilon_{2n-1}+\varepsilon_{2n},0}. \tag{4.29}$$

Furthermore, we also have

$$\sum_{s=\pm1} \mathrm{e}^{-s\frac{\pi\sqrt{-1}k}{r}} t_{k+s}^{k}(u_{2n-1}-u_0)^{\varepsilon_{2n-1}} t_k^{k+s}(u_{2n}-u_0)^{\varepsilon_{2n}}\Big|_{u_{2n}=u_{2n-1}-1-\frac{\pi\sqrt{-1}}{\epsilon}}$$
$$=\varepsilon_{2n}\,\mathrm{e}^{-\frac{\pi\sqrt{-1}}{2r}}\{k\}\delta_{\varepsilon_{2n-1}+\varepsilon_{2n},0}. \tag{4.30}$$

Thus, the normalization condition (4.6) in the XYZ Heisenberg antiferromagnet follows from (4.29), (4.30) and (4.27). □

From corollary 3 and proposition 4, we have the following theorem.

Theorem 5. *The XYZ correlation function $G_\sigma^{(n)}(\zeta)$ (4.18) solves the bootstrap equations (4.4)–(4.6).*

4.5. Second integral formula for the XYZ correlation functions

In this subsection we assume that $r \in \mathbb{Z}_{>1}$. The integral formula (4.18) remains the sum with respect to the local height variables $k, k_1, \ldots, k_{2n}$ in the cyclic SOS model. Thus, expression (4.18) consists of $2r \times 2^{2n}$ terms because of (4.19) and the condition $k_j = k_{j-1} \pm 1$ for $1 \leqslant j \leqslant 2n$.

Now we wish to present another integral solution to the bootstrap equations (4.4)–(4.6). Equation (4.22) shows that the sum with respect to k_{2n} with fixed $k, k_1, \ldots, k_{2n-1}$ can be performed to be in a simple form. In order to repeat this procedure with respect to $k_{2n-1}, \ldots, k_1$, we find that the number of integral variables should be greater than $2n$ for the $2n$-point correlation function.

Let $k_{2n} = k + 2n - 2l$ $(0 \leqslant l \leqslant 2n)$. Then the number of the set A is equal to $2n - l$, where

$$A := \{a | k_a = k_{a-1} + 1, 1 \leqslant a \leqslant 2n\}.$$

We further introduce the set of indices

$$A'' = A \sqcup \{2n+1, \ldots, (r+1)n + l\}.$$

Note that the number of the set A'' is equal to $(r+1)n$. Let $k_{2n+i} = k_{2n}+2i = k+2(n-l+i)$ for $1 \leqslant i \leqslant (r-1)n+l$. Then the meromorphic function $Q^{(n)}(w|\zeta)^{kk_1\cdots k_{2n}}$ with $k_{2n} = k+2n-2l$ is defined as follows:

$$Q^{(n)}(w|\zeta)^{kk_1\cdots k_{2n}} = \prod_{a\in A}\left\{v_a - u_a + \frac{1}{2} - k_a\right\}\left(\prod_{j=1}^{a-1}\left[v_a - u_j - \frac{1}{2}\right]\prod_{j=a+1}^{2n}\left[u_j - v_a - \frac{1}{2}\right]\right)$$
$$\times (-1)^{n-l}\prod_{a'=2n+1}^{(r+1)n+l}\frac{\left\{v_{a'} - u_0 + \frac{1}{2} - k_{a'}\right\}}{\left[v_{a'} - u_0 - \frac{1}{2}\right]}\left(\prod_{j=1}^{2n}\left[v_{a'} - u_j - \frac{1}{2}\right]\right)$$
$$\times \prod_{j=1}^{2n-1}\{k_j\}^{-1}\prod_{\substack{a,b\in A''\\ a<b}}[v_a - v_b + 1]^{-1}. \tag{4.31}$$

Set

$$\tilde{F}_\sigma^{(n)}(\zeta)^{kk_1\cdots k_{2n}} = \tilde{c}_n\prod_{1\leqslant j<k\leqslant 2n}\zeta_j^{\frac{r-1}{r}}g(z_j/z_k)\times\overline{\tilde{F}}_\sigma^{(n)}(\zeta)^{kk_1\cdots k_{2n}} \tag{4.32}$$

where $\tilde{c}_n$ is some constant, and

$$\overline{\tilde{F}}_\sigma^{(n)}(\zeta)^{kk_1\cdots k_{2n}} = \prod_{a\in A''}\oint_{C_a}\frac{\mathrm{d}w_a}{2\pi\sqrt{-1}w_a}\tilde{\Psi}_\sigma^{(n)}(w|\zeta)Q^{(n)}(w|\zeta)^{kk_1\cdots k_{2n}}. \tag{4.33}$$

Here, the kernel has the form

$$\tilde{\Psi}_\sigma^{(n)}(w|\zeta) = \tilde{\vartheta}_\sigma^{(n)}(w|\zeta)\prod_{a\in A''}\prod_{j=1}^{2n}x^{-\frac{(v_a-u_j)^2}{2r}}\psi\left(\frac{w_a}{z_j}\right)\prod_{1\leqslant j<k\leqslant 2n}x^{-\frac{(u_j-u_k)^2}{4r}}. \tag{4.34}$$

The function $\tilde{\vartheta}_\sigma^{(n)}(w|\zeta)$ is a function of $(r+1)n$ w and $2n$ ζ. The properties of $\tilde{\vartheta}_\sigma^{(n)}(w|\zeta)$ are the same as those of $\vartheta_\sigma^{(n)}(w|\zeta)$, but on the RHS of (3.14) the product with respect to a should be taken over A''.

The second integral formula of the XYZ correlation functions is as follows:

$$\tilde{G}_\sigma^{(n)}(\zeta) = (\sqrt{-1})^{-n}\sum_{k_0,k_1,\ldots,k_{2n}}t_{k_1}^{k_0}(u_1-u_0)\otimes\cdots\otimes t_{k_{2n}}^{k_{2n-1}}(u_{2n}-u_0)\tilde{F}_\sigma^{(n)}(\zeta)^{k_0k_1\cdots k_{2n}}. \tag{4.35}$$

This formula essentially solves the first two of the bootstrap equations. The R-matrix symmetry (4.4) evidently holds. The modified cyclicity

$$P_{12}\cdots P_{2n-12n}\tilde{G}_\sigma^{(n)}(\zeta', x^2\zeta_{2n}) = \sigma(-1)^{(r-1)n}\tilde{G}_\sigma^{(n)}(\zeta_{2n},\zeta') \tag{4.36}$$

can be proved in a similar manner as in proposition 2 and corollary 3.

The advantage of the second formula consists in the fact that the sum with respect to $k_1,\ldots,k_{2n}$ can be carried out. By repeating the similar calculation as (4.22) was derived, we find

$$\tilde{G}_\sigma^{(n)}(\zeta) = \prod_{a\in A''}\oint_{C_a}\frac{\mathrm{d}w_a}{2\pi\sqrt{-1}w_a}\tilde{\Psi}_\sigma^{(n)}(w|\zeta)\tilde{Q}^{(n)}(w|\zeta)$$

where

$$\tilde{Q}^{(n)}(w|\zeta)^{\varepsilon_1\cdots\varepsilon_{2n}} = \sum_{k=0}^{2r-1}(\sqrt{-1})^{-n+\sum_{j=0}^{2n}\varepsilon_j(k-2n-1+j)}(-1)^{(r+1)n}$$
$$\times\prod_{a=1}^{2n}\frac{[u_a-u_0]}{\left[v_a-u_0-\frac{1}{2}\right]}\left(\prod_{j=1}^{a-1}\left[v_a-u_j-\frac{1}{2}\right]\prod_{j=a+1}^{2n}\left[u_j-v_a-\frac{1}{2}\right]\right)$$
$$\times\prod_{i=1}^{(r-1)n}\frac{\left\{v_{2n+i}-u_0+\frac{1}{2}-k-2(n+i)\right\}}{\left[v_{2n+i}-u_0-\frac{1}{2}\right]}\left(\prod_{j=1}^{2n}\left[v_{2n+i}-u_j-\frac{1}{2}\right]\right)$$
$$\times\prod_{j=1}^{2n}f(u_j-u_0)h_{\varepsilon_j}(k+j+u_0+u_j-2v_j)\prod_{\substack{a,b\in A''\\ a<b}}[v_a-v_b+1]^{-1}. \tag{4.37}$$

Finally, we have to carry out the sum with respect to k of the following form:

$$F^{(n)}(u_0|\{u_j\}_{1\leqslant j\leqslant 2n}|\{v_a\}_{1\leqslant a\leqslant (r+1)n})^{\varepsilon} = \sum_{k=0}^{2r-1}(\sqrt{-1})^{k\sum_{j=0}^{2n}\varepsilon_j}\prod_{j=1}^{2n}h_{\varepsilon_j}(k+j+u_0+u_j-2v_j)$$
$$\times\prod_{i=1}^{(r-1)n}\left\{v_{2n+i}-u_0+\frac{1}{2}-k-2(n+i)\right\}. \tag{4.38}$$

This function $F^{(n)}$ has precise quasi-periodicities as a function of u_0, u_j $(1\leqslant j\leqslant 2n)$ and v_a $(1\leqslant a\leqslant (r+1)n)$. In particular, you can easily find the u_0-dependent part of $F^{(1)}$. Furthermore, if the limit $r\to 1$ is taken, the explicit expression for $n=1$ can be obtained as follows:

$$F^{(1)}(u_0|u_1,u_2|v_1,v_2)^{\varepsilon_1\varepsilon_2}$$
$$= \begin{cases} C_-\theta_2\left(\frac{u_1-u_2}{2}-v_1+v_2;\frac{\pi\sqrt{-1}}{\epsilon}\right)\theta_3\left(u_0+\frac{u_1+u_2}{2}-v_1-v_2;\frac{\pi\sqrt{-1}}{\epsilon}\right) & (\varepsilon_1\varepsilon_2<0)\\ C_+\theta_1\left(\frac{u_1-u_2}{2}-v_1+v_2;\frac{\pi\sqrt{-1}}{\epsilon}\right)\theta_4\left(u_0+\frac{u_1+u_2}{2}-v_1-v_2;\frac{\pi\sqrt{-1}}{\epsilon}\right) & (\varepsilon_1\varepsilon_2>0). \end{cases} \tag{4.39}$$

Here $C_\pm$ are some constants times exponential functions of u_0, u_1, u_2, v_1, v_2.

5. Concluding remarks

In this paper we have constructed two integral solutions to the bootstrap equations for the XYZ Heisenberg antiferromagnet. These solutions are expected to give the correlation functions of the XYZ model in the antiferromagnetic regime. The first solution is essentially the same as the formula given by Lashkevich and Pugai [9]. Lashkevich–Pugai's formula can be obtained from the correlation function in the RSOS-type model, by using the vertex–face correspondence. In order to avoid the pole resulting from $[k]$ in the denominator, $k\in\mathbb{Z}+\delta$ should be assumed with some real δ, and the limit $\delta\to 0$ should be taken after all calculation [9]. On the other hand, we constructed our first formula from the correlation function in the cyclic SOS model so that no such regularization was needed.

By the construction, both Lashkevich–Pugai's formula and our first formula for $2n$-point XYZ correlation function remain the sum with respect to the local height variables $k_0, k_1, \ldots, k_{2n}$ other than the n-fold integral. When $r\in\mathbb{Z}_{>1}$ we construct another $(r+1)n$-fold integral formula, in which the sum with respect to $k_1,\ldots,k_{2n}$ can be carried out. The structure of our second formula is quite similar to Shiraishi's formula [23], though the latter treated only the case $r = \frac{3}{2}$.

Concerning the second formula, we have not yet proved the normalization condition. The number of integral variables in the second formula is much greater than that in the first one, so that the recursion relation for the second one is not so simple. Nevertheless, we believe that the normalization condition holds in the second integral formula. Actually, the number of integral variables in the second formula is not minimal but redundant. It is sufficient for the number of integral variables N in the second formula to satisfy $n \leqslant N - n \equiv 0 \pmod r$. Thus, we can make an $N(n)$-fold integral formula, where $N(n) = n + mr$ for $(m-1)r < n \leqslant mr$ with some $m \in \mathbb{Z}_{>0}$. If we employ this 'minimal' integral solution, the normalization condition holds unless $n \equiv 1 \pmod r$ for the same reason as in the first formula. However, the validity of the recursion relation in the 'minimal' integral solution again becomes unclear for $n = mr + 1$ because $N(n) - N(n-1) = r + 1$.

Another future problem is to consider the bootstrap equations for Belavin's $\mathbb{Z}/n\mathbb{Z}$-symmetric model and to construct integral formulae of the correlation functions. We wish to address this problem in a separate paper.

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Appendix. Properties of the *R*-matrix

The main properties of the R-matrix of the eight-vertex model are the Yang–Baxter equation

$$R_{12}(\zeta_1/\zeta_2)R_{13}(\zeta_1/\zeta_3)R_{23}(\zeta_2/\zeta_3) = R_{23}(\zeta_2/\zeta_3)R_{13}(\zeta_1/\zeta_3)R_{12}(\zeta_1/\zeta_2) \tag{A.1}$$

where the subscript of the R-matrix denotes the spaces on which R nontrivially acts; the initial condition

$$R(1) = P \tag{A.2}$$

the unitarity relation

$$R_{12}(\zeta_1/\zeta_2)R_{21}(\zeta_2/\zeta_1) = 1 \tag{A.3}$$

the $\mathbb{Z}_2$-parity

$$R_{12}(-\zeta) = -\sigma_1^z R_{12}(\zeta)\sigma_1^z \tag{A.4}$$

and the crossing symmetries

$$R_{21}^{t_1}(\zeta_2/\zeta_1) = \sigma_1^x R_{12}(x^{-1}\zeta_1/\zeta_2)\sigma_1^x = -\sigma_1^y R_{12}(-x^{-1}\zeta_1/\zeta_2)\sigma_1^y. \tag{A.5}$$

In (A.4), (A.5) the shift $\zeta \mapsto -\zeta$ implies the one such that $u \mapsto u - \frac{\pi\sqrt{-1}}{\epsilon}$. The properties (A.2)–(A.5) hold if the normalization factor of the R-matrix satisfies the following relations:

$$\kappa(\zeta)\kappa(\zeta^{-1}) = 1 \qquad \kappa(x^{-1}\zeta^{-1}) = \kappa(\zeta). \tag{A.6}$$

Under this normalization the partition function per lattice site is equal to unity in the thermodynamic limit [12, 17].

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